Spectral Triples: Examples and Applications

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Contents

Chapter 1

Introduction

In these notes we try to motivate and develop some of the basic ideas and tools of spectral triples. Spectral triples play several related roles in noncommutative geometry. The first and most important is their role in index theory, and we focus on this as the main reason for looking at spectral triples.

The use of the Atiyah-Singer index theorem in identifying the obstructions to quantisation in gauge field theory known as anomalies is a paradigm for how index theory, and the geometric computation of indices, can help the modern physicist.

The second reason is the role of spectral triples as geometric spaces. Many spectral triples possess extra structure which enable us to do geometry by analogy with the spectral triples of classical manifolds. For this reason we will begin by looking at examples arising from the geometry of manifolds. Otherwise, however, we will not focus on the additional structures one may impose on a spectral triple. The interested reader can pursue these topics in [GVF, V], where very good accounts are presented.

Another physics reason for looking at spectral triples which we won't have time for is the role they are playing in physics, especially providing a noncommutative interpretation of the standard model of particle physics. The work by Bellissard using noncommutative geometry techniques to deduce the integrality of the quantum Hall current has also been influential.

If one was happy to simply accept the notion of spectral triple without motivation, then the study of spectral triples would devolve into some fancy functional analysis. This would belie the depth of the subject, its applications, and interpretations. Also, if one never left functional analysis, the range of examples would be almost nonexistent.

The examples arise from understanding how geometry, even very singular geometry, gives rise to the data defining a spectral triple. Even in highly noncommutative examples, one often proceeds by constructing analogues of structures which arise in differential or algebraic geometry. Frequently theorems and proofs are suggested by analogy with classical geometry.

To address the geometric fundamentals along with the index theoretic aspects, these notes focus on Dirac type operators on manifolds. A fair amount of differential geometry is assumed, but some points are recapped throughout. Also, we suppose that people know what a C^*-algebra is, and something of their representation theory on Hilbert spaces.

General references for some of the topics covered include:

Presentations of index theory on manifolds which adapt well to noncommutative geometry can be found in [BGV, HR, LM, G].

The basics of noncommutative geometry and spectral triples can be found in [GVF, Lan]. More sophisticated applications appear in [C0, C1] and [CMa].

Noncommutative algebraic topology, K-theory and K-homology, are beautifully presented in [HR]. More introductory books on K-theory are [RLL] and [WO]. For K-theory of spaces see [AK]. Noncommutative differential topology is cyclic homology and cohomology. The description in [C1] is excellent, and further information is available in [L].

A wonderful exposition of the intertwining of spectral triples and the index theorem in noncommutative geometry is [H] (also available on Nigel Higson's website).

These texts all have references to original papers which are also important to read.

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Conventions

Hilbert spaces are separable and complex. The bounded linear operators on a Hilbert space $\mathcal{H}$ are denoted $\mathcal{B}(\mathcal{H})$. The ideal of compact operators on $\mathcal{H}$ is denoted $\mathcal{K}(\mathcal{H})$. The Calkin algebra is denoted $\mathcal{Q}(\mathcal{H}) := \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$.

C^*-algebras are separable and complex, and almost always unital in these notes.

M will always be an n-dimensional compact oriented manifold. Usually we will suppose that it has a Riemannian metric g.

X is always a compact Hausdorff space, $C(X)$ the C^*-algebra of continuous functions on X.

We let $\Lambda^*M := \Lambda^*T^*M = \oplus_{k=0}^n \Lambda^k T^*M$ denote the bundle of exterior differential forms, and $\Gamma(\Lambda^*M)$ the smooth sections of Λ^*M.

Chapter 2

Preliminaries and the first example

2.1 The Fredholm index

The roots of noncommutative geometry lie in index theory. The central classical problem here is to compute an integer, called the index, associated with certain special operators on manifolds, the elliptic pseudodifferential operators. The solution to this problem was provided by Atiyah and Singer in the 1960's, and we will discuss numerous examples and what the theorem says later. In this section, we will just review what the Fredholm index is. The discussion of the index in [LM] is quite good and set in the context of the Atiyah-Singer index theorem.

Definition 2.1. *Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces and $F : \mathcal{H}_1 \to \mathcal{H}_2$ a bounded linear operator. We say that F is Fredholm if*

1) $\mathrm{range}(F)$ *is closed in $\mathcal{H}_2$, and*

2) $\ker(F)$ *is finite dimensional, and*

3) $\mathrm{coker}(F) := \mathcal{H}_2/\mathrm{range}(F)$ *is finite dimensional. If F is Fredholm we define*

$$\mathrm{Index}(F) = \dim\ker(F) - \dim\mathrm{coker}(F).$$

Example 1. The simplest Fredholm operator is the shift operator $S : l^2(\mathbb{N}) \to l^2(\mathbb{N})$. This is defined by

$$S\sum_{i=1}^{\infty} a_i e_i = \sum_{i=1}^{\infty} a_i e_{i+1}, \quad a_i \in \mathbb{C}.$$

The range of S is codimension 1, and so is closed. The kernel of S is $\{0\}$, and so

$$\mathrm{Index}(S) = \dim\ker(S) - \dim\mathrm{coker}(S) = 0 - 1 = -1.$$

Example 2. If $F : \mathcal{H} \to \mathcal{H}$ is a Fredholm operator and F is self-adjoint, then $\mathrm{Index}(F) = 0$. This is because in general $\mathrm{coker}(F) = \ker(F^*)$.

Definition 2.2. *A bounded linear operator $T : \mathcal{H}_1 \to \mathcal{H}_2$ is called compact if T maps any bounded sequence $\{\xi_k\}_{k\geq 0} \in \mathcal{H}_1$ to a sequence $\{T\xi_k\}_{k\geq 0} \in \mathcal{H}_2$ with a convergent subsequence. Equivalently, T is compact if it is the norm limit of a sequence of finite rank operators.*

The set of compact operators mapping a Hilbert space $\mathcal{H}$ to itself is denoted $\mathcal{K}(\mathcal{H})$ and is an ideal in $\mathcal{B}(\mathcal{H})$. It is the only norm closed ideal in $\mathcal{B}(\mathcal{H})$. We introduce compact operators at this point because they allow us to give a characterisation of Fredholm operators.

Proposition 2.3. *Let $F : \mathcal{H}_1 \to \mathcal{H}_2$ and $S : \mathcal{H}_2 \to \mathcal{H}_1$ be bounded linear operators such that $FS - Id_{\mathcal{H}_2}$ and $SF - Id_{\mathcal{H}_1}$ are compact operators (on $\mathcal{H}_2$ and $\mathcal{H}_1$ respectively). Then F, S are Fredholm operators. The converse is also true.*

Remark Given F and S as in the Proposition, S is said to be a parametrix or approximate inverse for F, and vice versa.

Thus the Fredholm operators are precisely those which are invertible modulo compact operators. If we denote by

$$q : \mathcal{B}(\mathcal{H}) \to \mathcal{Q}(\mathcal{H}) := \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$$

the quotient map onto the Calkin algebra $\mathcal{Q}(\mathcal{H})$, the image of the Fredholm operators $F : \mathcal{H} \to \mathcal{H}$ lies in the group of invertibles of $\mathcal{Q}(\mathcal{H})$.

Exercise Show that if $F, S : \mathcal{H} \to \mathcal{H}$ are both Fredholm operators, then $FS : \mathcal{H} \to \mathcal{H}$ is also Fredholm.

We now summarise the important properties of the index of Fredholm operators.

Theorem 2.4. *Let $\mathcal{F}$ denote the set of Fredholm operators on a fixed Hilbert space $\mathcal{H}$, and let $\pi_0(\mathcal{F})$ denote the (norm) connected components of $\mathcal{F}$. The index is locally constant on $\mathcal{F}$ and induces a bijection*

$$\mathrm{Index} : \pi_0(\mathcal{F}) \to \mathbb{Z}. \tag{2.1}$$

Moreover, the index satisfies

$$\mathrm{Index}(F^*) = -\,\mathrm{Index}(F), \qquad \mathrm{Index}(FS) = \mathrm{Index}(F) + \mathrm{Index}(S),$$

and so the induced map (2.1) is a group isomorphism.

In particular, any two operators with the same index lie in the same connected component of $\mathcal{F}$ and the index is constant on these components (which are open in the norm topology). It is also worth observing that if F is Fredholm and T is compact then $F + T$ is Fredholm and

$$\mathrm{Index}(F + T) = \mathrm{Index}(F).$$

Thus the index is constant under compact perturbations and sufficiently small norm perturbations. This gives us strong invariance properties for the index. For instance, if $\{F_t\}_{t\in[0,1]}$ is a norm continuous path of Fredholm operators, then $\mathrm{Index}(F_t)$ is constant.

By considering operators on manifolds which give rise to Fredholm operators on Hilbert space, we will be able to construct invariants of the underlying manifold. Amazingly, we can frequently extend this same strategy to noncommutative spaces.

2.2 Clifford Algebras

Clifford algebras play a central role in the construction and analysis of many important geometric operators on manifolds. It is worth introducing them early, as it will streamline much of what we will do. Basic references for this material include [ABS, BGV, GVF, LM].

Let V be a finite dimensional real vector space, and $(\cdot|\cdot) : V \times V \to \mathbb{R}$ a bilinear inner product, so for $u, v, w \in V$ and $\lambda \in \mathbb{R}$

$$(v|w) = (w|v), \quad (\lambda v|w) = \lambda(v|w), \quad (v + u|w) = (v|w) + (u|w).$$

We suppose also that the inner product is nondegenerate so that $(v|v) = 0 \Rightarrow v = 0$.

Definition 2.5. *The Clifford algebra $Cliff(V, (\cdot|\cdot))$ (we write $Cliff(V)$ when $(\cdot|\cdot)$ is understood) is the unital associative algebra over $\mathbb{R}$ generated by all $v \in V$ and $\lambda \in \mathbb{R}$ subject to*

$$v \cdot w + w \cdot v = -2(v|w)Id_{Cliff(V)}.$$

There are several other ways to define the Clifford algebra, but this is enough for us.

Observe that if v, w are orthogonal, they anticommute. Indeed, if the inner product were zero, we would simply wind up with the exterior algebra of V. In fact

Lemma 2.6. *The two algebras Λ^*V and $Cliff(V)$ are linearly isomorphic (although not isomorphic as algebras).*

Proof. We define the map $m : \Lambda^*V \to Cliff(V)$ by

$$m(v_1 \wedge v_2 \wedge \cdots \wedge v_k) = v_1 \cdot v_2 \cdot \cdots \cdot v_k.$$

We leave it as an exercise to check this is an isomorphism. □

Hence we can regard the Clifford algebra as the exterior algebra with a deformed product. In [BGV] the map m is called a quantization map.

Exercise Write down the inverse to the isomorphism m.

Most of the time, we actually want the complexification of the Clifford algebra, $\mathbb{C}liff(V) = Cliff(V) \otimes \mathbb{C}$. This is more compatible with working on complex Hilbert space, and the complexifications are actually simpler in many ways.

Exercise Show that

$$\mathbb{C}liff(\mathbb{R}) = \mathbb{C} \oplus \mathbb{C}, \qquad \mathbb{C}liff(\mathbb{R}^2) = M_2(\mathbb{C}).$$

More generally we have

$$\mathbb{C}liff(\mathbb{R}^k) = \begin{cases} M_{2^{(k-1)/2}}(\mathbb{C}) \oplus M_{2^{(k-1)/2}}(\mathbb{C}) & k \text{ odd} \\ M_{2^{k/2}}(\mathbb{C}) & k \text{ even} \end{cases}.$$

A useful characterisation is the following.

Lemma 2.7. *If A is a complex unital associative algebra and $c : V \to A$ is a linear map satisfying*

$$c(v)c(w) + c(w)c(v) = -2(v|w)1_A,$$

for all $v, w \in V$, then there is a unique algebra homomorphism $\tilde{c} : \mathbb{C}liff(V) \to A$ extending c. The analogous statement is true for real Clifford algebras.

There is a special element inside the Clifford algebra called the (complex) volume form. Suppose V is n-dimensional and let $e_1, e_2, \ldots, e_n$ be an orthonormal basis of V. Define

$$\omega_{\mathbb{C}} = i^{[(n+1)/2]} e_1 \cdot e_2 \cdot \cdots \cdot e_n.$$

Then $\omega_{\mathbb{C}}^2 = 1$ and for all $v \in V$ we have

$$v \cdot \omega_{\mathbb{C}} = (-1)^{n-1} \omega_{\mathbb{C}} \cdot v.$$

If we give the Clifford algebra the adjoint

$$(\lambda e_1 \cdots e_k)^* = (-1)^k \bar{\lambda} e_k \cdots e_1$$

then the Clifford algebra becomes a C^*-algebra, and $\omega_{\mathbb{C}} = \omega_{\mathbb{C}}^*$.

It is useful in what follows to express the Clifford multiplication by $v \in V$ in terms of the exterior algebra. To do this, we need to recall the **interior product** on $\Lambda^* V$. For $v \in V$ and $v_1 \wedge v_2 \wedge \cdots \wedge v_k$ we define

$$v \llcorner (v_1 \wedge v_2 \wedge \cdots \wedge v_k) = \sum_{i=1}^{k} (-1)^{i+1} (v_i|v) v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_k,$$

where $\hat{}$ denotes omission. The interior product satisfies

$$v \llcorner (\varphi \wedge \psi) = (v \llcorner \varphi) \wedge \psi + (-1)^k \varphi \wedge (v \llcorner \psi), \quad \varphi \in \Lambda^k V,$$

and so $v \llcorner \circ v \llcorner = 0$. Also observe that $\bar{v} \llcorner$ is the adjoint of $v \wedge$ for the inner product on $\Lambda^* V$ (see below), where $\bar{v}$ is the complex conjugate vector. Then under the isomorphism $\Lambda^* V \cong Cliff(V)$, we have for $v \in V$ and $\varphi \in Cliff(V)$

$$v \cdot \varphi = v \wedge \varphi - v \llcorner \varphi. \tag{2.2}$$

Since for all $\varphi \in \Lambda^* V$ we have

$$v \wedge (w \llcorner \varphi) + w \llcorner (v \wedge \varphi) = (v|w)\varphi$$

one can check that the action of V on $\Lambda^* V$ defined by the formula in Equation (2.2) satisfies

$$v \cdot w + w \cdot v = -2(v|w) Id_V$$

and so extends to an action of $Cliff(V)$ on $\Lambda^* V$. Similarly right multiplication by v gives

$$\varphi \cdot v = (-1)^k (v \wedge \varphi + v \llcorner \varphi), \quad \varphi \in \Lambda^k V.$$

By associativity, the left and right actions of $Cliff(V)$ commute with one another, so $\Lambda^* V$ carries two commuting actions of the Clifford algebra. The complexification of $\Lambda^* V$ also carries commuting representations of the complexified Clifford algebra.

If V has an inner product, as we suppose, then we define a sesquilinear (bilinear if V is real) map

$$(\cdot|\cdot)^k : \Lambda^k V \times \Lambda^k V \to \mathbb{R}$$

by

$$(u_1 \wedge \cdots \wedge u_p | v_1 \wedge \cdots \wedge v_p)^k := \det \begin{pmatrix} (u_1|v_1) & \cdots & (u_1|v_p) \\ \vdots & \ddots & \vdots \\ (u_p|v_1) & \cdots & (u_p|v_p) \end{pmatrix}.$$

Choose an oriented orthonormal basis $e_1, ..., e_n$ of V and let $\sigma = e_1 \wedge \cdots \wedge e_n$. For $\lambda \in \Lambda^k V$, the map

$$\lambda \wedge \cdot : \Lambda^{n-k} V \to \Lambda^n V,$$

is linear, and as $\Lambda^n V$ is one dimensional, there exists a unique $f_\lambda \in Hom(\Lambda^{n-k} V, \mathbb{R})$ such that

$$\lambda \wedge \mu = f_\lambda(\mu)\sigma, \quad \forall \mu \in \Lambda^{n-k} V.$$

As $\Lambda^{n-k} V$ is an inner product space, every such linear form is given by the inner product with a fixed element of $\Lambda^{n-k} V$, which in this case depends on λ. Denote this element by $*\lambda$. So

$$f_\lambda(\mu) = (\mu | *\lambda)^{n-k},$$

and

$$\lambda \wedge \mu = (\mu | *\lambda)^{n-k} \sigma, \quad \forall \mu \in \Lambda^{n-k} V.$$

The map

$$* : \Lambda^k V \to \Lambda^{n-k} V, \qquad \lambda \longrightarrow *\lambda$$

is called the **Hodge Star Operator**.

Lemma 2.8. *If V has a positive definite inner product, and $\lambda, \mu \in \Lambda^k V$, then*

$$*(*\lambda) = (-1)^{k(n-k)} \lambda,$$

$$\lambda \wedge *\mu = \mu \wedge *\lambda = (\lambda|\mu)^k \sigma.$$

This discussion of actions of the Clifford algebra on $\Lambda^* V$ and the Hodge star operator all makes sense for real Clifford algebras. Now in general, $\omega_{\mathbb{C}}$ is not an element of the real Clifford algebra (supposing V to be the complexification of a real vector space). Nevertheless, when it is in the real Clifford algebra we will see that $\omega_{\mathbb{C}}$ and the Hodge star operator are closely related.

All of this continues to make sense on a manifold M. Here we consider the vector bundle $\Lambda^* M$ and the sections $\Gamma(\Lambda^* M)$ with all the above operations defined pointwise. Similarly, we let $\mathbb{C}liff(M)$ denote the sections of the bundle of algebras $\mathbb{C}liff(T^*M, g)$, where g is a Riemannian inner product on T^*M.

2.3 The Hodge-de Rham operator

We build our first example out of ingredients immediately to hand. Basic references for all this material include [BGV, LM]. We let $L^2(\Lambda^* M, g)$ be the Hilbert space completion of the exterior bundle $\Lambda^* T^*_{\mathbb{C}} M$ with respect to the inner product

$$\langle \omega, \rho \rangle_g = \int_M \omega \wedge *\overline{\rho}.$$

Here $*$ is the Hodge $*$-operator, described in the previous Section.

This inner product ensures that forms of different degrees are orthogonal, and is positive definite since

$$\omega \wedge *\bar{\omega} = (\omega|\bar{\omega}) dvol.$$

The exterior derivative d extends to a closed unbounded operator on $L^2(\Lambda^* M, g)$, [HR, Lemma 10.2.1]. We let d^* be the adjoint of the exterior derivative with respect to this inner product. We let $\mathcal{D} = d + d^*$, and call this the **Hodge-de Rham operator**. This operator is formally self-adjoint (and so symmetric) and so by [HR, Corollary 10.2.6], extends to a self-adjoint operator on $L^2(\Lambda^* M, g)$.

2.3.1 The symbol and ellipticity

Before analysing this example any further, we need to recall the principal symbol of a differential operator $D : \Gamma(E) \to \Gamma(F)$ between sections of vector bundles E, F over M. The principal symbol σ_D associates to each $x \in M$ and $\xi \in T^*_x M$ a linear map $\sigma_D(x, \xi) : E_x \to F_x$ defined as follows. If D is order m and in local coordinates we have

$$D = \sum_{|\alpha| \leq m} M_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}, \qquad \xi = \sum \xi_k dx^k \in T^*_x M$$

then

$$\sigma_D(x, \xi) = \sum_{|\alpha| = m} M_\alpha(x) \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}.$$

(There is usually a factor of i^m in the symbol to make it compatible with the Fourier transform and operators on real bundles, but we won't worry). This local coordinate description can be pasted together to give a globally defined map

$$\sigma_D : T^* M \to Hom(E, F).$$

Lemma 2.9. *[HR, Chapter 10] Let D be a first order differential operator on a smooth compact manifold M. Then for $f \in C^\infty(M)$*

$$[D, f] = \sigma_D(df).$$

Proof. Just compute. □

Let's apply this result to the Hodge-de Rham operator. First observe that

$$\begin{aligned}(d(f\omega)|\rho) &= (df\wedge\omega|\rho) + (fd\omega|\rho)\\ &= (\omega|d\bar{f}\llcorner\rho) + (d\omega|\bar{f}\rho) \quad \text{since} \quad (v\wedge)^* = \bar{v}\llcorner\\ &= (\omega|d\bar{f}\llcorner\rho) + (\omega|d^*(\bar{f}\rho))\end{aligned}$$

So

$$(\omega|d^*(\bar{f}\rho)) = (\omega|\bar{f}d^*\rho) - (\omega|d\bar{f}\llcorner\rho).$$

Since this is true for all forms ω,ρ and all smooth functions f, we deduce that for all forms φ and functions f

$$d^*(f\varphi) = fd^*\varphi - df\llcorner\varphi.$$

Now for $\varphi\in\Gamma(\Lambda^*M)$ we can compute

$$[d+d^*, f]\varphi = df\wedge\varphi + fd\varphi - df\llcorner\varphi + fd^*\varphi - fd\varphi - fd^*\varphi = df\wedge\varphi - df\llcorner\varphi.$$

Hence the principal symbol of $d+d^*$ is given by the left Clifford action on Λ^*M. In particular, for all $f\in C^\infty(M)$, the commutator $[d+d^*, f]$ extends to a bounded operator on $L^2(\Lambda^*M, g)$.

Before moving on, let me quote another useful result from the theory of pseudodifferential operators.

Lemma 2.10. *Suppose that $Q,P:\Gamma(E)\to\Gamma(E)$ are two (pseudo)differential operators on the same vector bundle of orders $q,p\geq 0$ respectively. If their principal symbols commute, then*

$$\textit{order}([Q,P]) \leq q+p-1.$$

Since multiplication by f has principal symbol fId, it commutes with any endomorphism, and so for a first order operator like $d+d^*$, the commutator is order zero, namely an endomorphism.

Definition 2.11. *Let $P:\Gamma(E)\to\Gamma(F)$ be a differential operator with principal symbol $\sigma_P: T^*M\to Hom(E,F)$. If for all $x\in M$ and $0\neq\xi\in T^*_xM$ we have $\sigma_P(x,\xi)$ is an isomorphism, we call P an* **elliptic** *operator.*

Since $\xi\cdot\xi\cdot = -\|\xi\|^2$ where the norm is the one coming from the inner product in T^*_xM, we see that the Hodge-de Rham operator has invertible principal symbol for all $\xi\neq 0$, and so $d+d^*$ is elliptic.

2.3.2 Ellipticity and Fredholm properties

Observe also that if we define $\gamma:\Gamma(\Lambda^*M)\to\Gamma(\Lambda^*M)$ by

$$\gamma(\omega) = (-1)^k\omega, \quad \omega\in\Lambda^kM$$

then since both d and d^* change the degree of a form by one we have

$$\gamma(d+d^*) = -(d+d^*)\gamma.$$

Since $\gamma^2 = 1$, γ has ± 1 eigenvalues (on $\Lambda^{even}M$ and $\Lambda^{odd}M$) and we can write

$$\Lambda^* M = \Lambda^{even} M \oplus \Lambda^{odd} M, \quad \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad d + d^* = \begin{pmatrix} 0 & (d+d^*)^- \\ (d+d^*)^+ & 0 \end{pmatrix}.$$

The differential operator $(d+d^*)^+ : \Gamma(\Lambda^{even}M) \to \Gamma(\Lambda^{odd}M)$ has adjoint $(d+d^*)^- : \Gamma(\Lambda^{odd}M) \to \Gamma(\Lambda^{even}M)$.

You can probably guess that I am going to tell you that this operator $(d+d^*)^+ : L^2(\Lambda^{even}M) \to L^2(\Lambda^{odd}M)$ is Fredholm and the index means something really interesting.

Well, yes, but we need to be a little careful. The first problem is that $d + d^*$ is not a bounded operator on $L^2(\Lambda^* M, g)$, and so the definition of Fredholm doesn't even make sense as it stands. Here is the standard resolution of the problem.

Definition 2.12. *Let M be a compact oriented n-dimensional Riemannian manifold. For $s \geq 0$, define*

$$L^2_s(\Lambda^* M, g) = \{\xi \in L^2(\Lambda^* M, g) : (1+\Delta)^{s/2}\xi \in L^2(\Lambda^* M, g)\},$$

where $\Delta = (d+d^*)^2$ is the Hodge Laplacian. Then $L^2_s(\Lambda^* M, g)$ is a Hilbert space for the inner product

$$\langle \xi, \eta \rangle_s := \langle \xi, \eta \rangle + \langle (1+\Delta)^{s/2}\xi, (1+\Delta)^{s/2}\eta \rangle$$

and we call this the s-th Sobolev space.

Remark We can do this for any vector bundle by choosing a connection ∇, and defining the connection Laplacian $\nabla^*\nabla$ which is positive. Other methods include a Fourier definition, and interpolation between integer Sobolev spaces.

The point of Sobolev spaces for us is the following easy proposition.

Proposition 2.13. *A differential operator $D : \Gamma(\Lambda^* M) \to \Gamma(\Lambda^* M)$ of order $m \geq 0$ extends to a bounded operator $D : L^2_s(\Lambda^* M) \to L^2_{s-m}(\Lambda^* M)$ for all $s \geq m$.*

What we would like to do is define the index of $(d+d^*)^+$ to be the index of this operator from $L^2_s \to L^2_{s-1}$. However, we need to know that the index is independent of s.

Theorem 2.14. *Let $P : \Gamma(E) \to \Gamma(F)$ be an elliptic (differential) operator of order $m \geq 0$ on M. Then*

1) For any open set $U \subset M$ and any $\xi \in L^2_s(E)$,

$$P\xi|_U \in C^\infty \Rightarrow \xi|_U \in C^\infty \tag{2.3}$$

2) For each $s \geq m$, P extends to a Fredholm operator $P : L^2_s(E) \to L^2_{s-m}(F)$ whose index is independent of s.

3) For each $s \geq m$ there is a constant C_s such that

$$\|\xi\|_s \leq C_s(\|\xi\|_{s-m} + \|P\xi\|_{s-m}) \qquad \text{Elliptic estimate} \tag{2.4}$$

for all $\xi \in L^2_s(E)$. Hence the norms $\|\cdot\|_s$ and $\|\cdot\|_{s-m} + \|P\cdot\|_{s-m}$ on $L^2_s(E)$ are equivalent.

The key to proving this theorem is the elliptic estimate. Once this is proved, the rest can be deduced reasonably simply.

So the index can be defined in a sensible way, but can it be related to the index of a bounded linear operator on $L^2(\Lambda^* M, g)$ without all this Sobolev space stuff? Do we want to do that? Would we learn any more? Answers: Yes, yes, yes.

Proposition 2.15. *Let $\mathcal{D}$ be a self-adjoint first order elliptic differential operator on M. Then the densely defined operator $(1+\mathcal{D}^2)^{-1/2} : L^2(\Lambda^* M, g) \to L^2(\Lambda^* M, g)$ is bounded and extends to a compact operator on $L^2(\Lambda^* M, g)$. Hence the operator $\mathcal{D}(1+\mathcal{D}^2)^{-1/2}$ is self-adjoint Fredholm.*

Proof. The operator $(1+\mathcal{D}^2)^{-1/2}$ maps $L^2 = L^2_0$ onto L^2_1, and is of norm (at most) one. The inclusion of L^2_1 into L^2_0 is a compact linear operator by the Rellich Lemma, and so $(1+\mathcal{D}^2)^{-1/2} : L^2 \to L^2$ is a compact operator.

The second statement follows because

$$\left(\mathcal{D}(1+\mathcal{D}^2)^{-1/2}\right)^2 = \mathcal{D}^2(1+\mathcal{D}^2)^{-1} = 1-(1+\mathcal{D}^2)^{-1}$$

and so $\mathcal{D}(1+\mathcal{D}^2)^{-1/2}$ has a parametrix (itself), and so is Fredholm. □

These are the key tools required to show that if we can split $\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}^- \\ \mathcal{D}^+ & 0 \end{pmatrix}$ like the $d+d^*$ operator, the index of $\mathcal{D}^+(1+\mathcal{D}^2)^{-1/2}$ equals that of all the closed extensions of $\mathcal{D}$ on Sobolev spaces. We will do this shortly in a more general context.

Before discussing the index of $(d+d^*)^+$, we quote one further result about elliptic differential operators.

Theorem 2.16. *[LM, Thm 5.5] Let $P : \Gamma(E) \to \Gamma(E)$ be an elliptic self-adjoint differential operator over a compact Riemannian manifold. Then there is an L^2-orthogonal direct sum decomposition*

$$\Gamma(E) = \ker P \oplus \text{Image } P.$$

2.3.3 The index of the Hodge-de Rham operator

To work out the index of $(d+d^*)^+ = \frac{1}{4}(1-\gamma)(d+d^*)(1+\gamma)$, we are going to need a little more machinary. Let $\Delta = (d+d^*)^2$ be the Laplacian on forms, and observe $\Delta = dd^* + d^*d$. Then $\text{Image}(\Delta) = \text{Image}(d) + \text{Image}(d^*)$ and so by Theorem 2.16

Proposition 2.17 (The Hodge Decomposition Theorem)**.** *Let M be a compact oriented Riemannian manifold, and let $\mathbf{H}^p$ denote the kernel of $\Delta = (d+d^*)^2$ on p-forms. Then there is an L^2-orthogonal direct sum decomposition*

$$\Gamma(\Lambda^p M) = \mathbf{H}^p \oplus \text{Image}(d) \oplus \text{Image}(d^*), \quad p = 0, ..., n. \tag{2.5}$$

In particular, there is an isomorphism

$$\mathbf{H}^p \cong H^p_{dR}(M; \mathbb{R}), \quad p = 0, ..., n,$$

where $H^p_{dR}(M; \mathbb{R})$ denotes the p-th de Rham cohomology group.

Proof. The first statement follows directly from Theorem 2.16. For the second, we observe that Equation (2.5) says

$$\mathbf{H}^p \oplus \mathrm{Image}(d) = \mathrm{coker}\, d^* = \ker d.$$

Hence

$$H^p_{dR}(M;\mathbb{R}) = \frac{\ker d}{\mathrm{Image}(d)} = \mathbf{H}^p.$$

□

It does not take long now to figure out

Theorem 2.18. *The index of* $(d + d^*)^+$ *is*

$$\begin{aligned} \mathrm{Index}(d + d^*)^+ &= \sum_{k=0}^{n} (-1)^k \dim H^k_{dR}(M;\mathbb{R}) \\ &= \chi(M) = \text{the Euler characteristic of } M \\ &= \text{a homotopy invariant of } M \\ &= \text{independent of the metric } g. \end{aligned} \tag{2.6}$$

These index calculations depend on the analysis of pseudodifferential operators, which we have omitted. In particular, it is the pseudodifferential machinery which allows us to see that for P elliptic, $Pu \in C^\infty \Rightarrow u \in C^\infty$. From there it is not hard to see that the kernel and cokernel of $(d + d^*)^+$ consist of smooth sections, and the index is independent of s.

For an introduction to this pseudodifferential theory, see [LM] or [G]. Alternatively, one can develop an abstract pseudodifferential calculus for spectral triples, and deduce the same results: we have included a brief outline of this calculus in Section 5.4.

2.4 The definition of a spectral triple

Definition 2.19. *A spectral triple* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ *is given by a* $*$*-algebra* $\mathcal{A}$ *represented on a Hilbert space* $\mathcal{H}$

$$\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}), \quad \pi \text{ is a } * -\text{homomorphism}$$

along with a densely defined self-adjoint (typically unbounded) operator

$$\mathcal{D} : \text{dom } \mathcal{D} \subset \mathcal{H} \to \mathcal{H}.$$

We require that

1) For all $a \in \mathcal{A}$, $\pi(a)\text{dom } \mathcal{D} \subset \text{dom } \mathcal{D}$ *and the densely defined operator* $[\mathcal{D}, \pi(a)] := \mathcal{D}\pi(a) - \pi(a)\mathcal{D}$ *is bounded (and so extends to a bounded operator on all of* $\mathcal{H}$ *by continuity),*

2) For all $a \in \mathcal{A}$ *the operator* $\pi(a)(1 + \mathcal{D}^2)^{-1/2}$ *is a compact operator.*

If in addition there is an operator $\gamma \in \mathcal{B}(\mathcal{H})$ *with* $\gamma = \gamma^*$, $\gamma^2 = 1$, $\mathcal{D}\gamma + \gamma\mathcal{D} = 0$, *and for all* $a \in \mathcal{A}$ $\gamma\pi(a) = \pi(a)\gamma$, *we call the spectral triple* **even** *or* **graded**. *Otherwise it is* **odd** *or* **ungraded**.

Remark This seems like an unwieldy definition. There are numerous ingredients, the unbounded operator makes things technically tricky, and there is a lot to check. This is true. However, the principal gains are that this is actually the structure one encounters naturally when doing index theory, and it is probably the easiest framework in which to compute. Hopefully we will see all this as the course progresses.

Remark We will nearly always dispense with the representation π, treating $\mathcal{A}$ as a subalgebra of $\mathcal{B}(\mathcal{H})$.

CONVENTION: Unless explicitly mentioned otherwise, all spectral triples will be unital, that is the algebra $\mathcal{A}$ is unital; i.e. there is $1 \in \mathcal{A}$ such that $1a = a1 = a$, $1^* = 1$. This implies in particular that $(1+\mathcal{D}^2)^{-1/2}$ is a compact operator.

Example 3. The Hodge-de Rham triple $(C^\infty(M), L^2(\Lambda^*, g), d + d^*)$ of an oriented compact manifold. It is always even, being graded by the degree of forms modulo 2.

Example 4. Let $\mathcal{H} = L^2(S^1)$, $\mathcal{A} = C^\infty(S^1)$ and

$$\mathcal{D} = \frac{1}{i}\frac{d}{d\theta}$$

where we are using local coordinates to define $\mathcal{D}$. This is an odd spectral triple, as a little Fourier theory will reveal. (**Exercise**)

2.4.1 Index properties of spectral triples

First we show that the '$\mathcal{D}$' of a spectral triple has a well-defined index.

Lemma 2.20. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple. Then $\mathcal{D}$ is unbounded Fredholm.*

Proof. By this we mean that $\mathcal{D}$ is a (bounded) Fredholm operator from the Hilbert space $\mathcal{H}_1 = \{\xi \in \mathcal{H} : \mathcal{D}\xi \in \mathcal{H}\}$ with the inner product

$$\langle \xi, \eta \rangle_1 := \langle \xi, \eta \rangle + \langle \mathcal{D}\xi, \mathcal{D}\eta \rangle$$

to the Hilbert space $\mathcal{H}$. To see this, one first checks that $\mathcal{D} : \mathcal{H}_1 \to \mathcal{H}$ is bounded (**Exercise**) and then produces an inverse up to compacts. Such an approximate inverse (parametrix) is given by

$$\mathcal{D}(1+\mathcal{D}^2)^{-1} : \mathcal{H} \to \mathcal{H}_1$$

since

$$\mathcal{D} \cdot \mathcal{D}(1+\mathcal{D}^2)^{-1} = 1 - (1+\mathcal{D}^2)^{-1}.$$

Exercise: Fill in the details of this proof. □

Since $\mathcal{D}$ is self-adjoint, it has zero index, but when $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is even, or graded, we also have

$$\mathcal{D}^+ = \frac{1-\gamma}{2}\mathcal{D}\frac{1+\gamma}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & \mathcal{D}^- \\ \mathcal{D}^+ & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{D}^+ : \mathcal{H}_1^+ \to \mathcal{H}^-.$$

For an even spectral triple, this is the operator of interest, and it too is Fredholm since $\mathcal{D}^+\mathcal{D}^-(1+\mathcal{D}^2)^{-1}$ is 'almost' the identity on $\mathcal{H}_-$. Since $\mathcal{D}$ is unbounded, it is not Fredholm in the strictest sense of the word, and we need to check that the index is well-defined.

Definition 2.21. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple. For $s \geq 0$ define $\mathcal{H}_s = \{\xi \in \mathcal{H} : (1 + \mathcal{D}^2)^{s/2}\xi \in \mathcal{H}\}$. With the inner product*

$$\langle \xi, \eta \rangle_s := \langle \xi, \eta \rangle + \langle (1 + \mathcal{D}^2)^{s/2}\xi, (1 + \mathcal{D}^2)^{s/2}\eta \rangle,$$

$\mathcal{H}_s$ is a Hilbert space. Finally let

$$\mathcal{H}_\infty := \bigcap_{s \geq 0} \mathcal{H}_s = \bigcap_{s \geq 0} \text{dom } (1 + \mathcal{D}^2)^{s/2}.$$

Corollary 2.22. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be an even spectral triple with grading γ. Write $\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}^- \\ \mathcal{D}^+ & 0 \end{pmatrix}$, and let $\mathcal{D}_s^+$ be the restriction $\mathcal{D}^+ : \mathcal{H}_s \to \mathcal{H}_{s-1}$ where $\mathcal{H}_0 = \mathcal{H}$. For all $s \geq 1$ we have*

$$\text{Index}(\mathcal{D}_s^+) = \text{Index}(\mathcal{D}^+) = \text{Index}(\mathcal{D}^+(1 + \mathcal{D}^2)^{-1/2}),$$

where the middle index is of $\mathcal{D}^+ : \mathcal{H}_1^+ \to \mathcal{H}^-$ and the last is the index of $\mathcal{D}^+(1 + \mathcal{D}^2)^{-1/2} : \mathcal{H}^+ \to \mathcal{H}^-$.

Proof. Suppose that $\mathcal{D}\xi = \lambda\xi$, so that ξ is an eigenvector. Then, since $\xi \in \text{Dom}\mathcal{D}$, we see that $\xi \in \mathcal{H}_\infty$. So all the eigenvectors of $\mathcal{D}$ lie in $\mathcal{H}_\infty$. In particular, if $\mathcal{D}\xi = 0$, $\xi \in \mathcal{H}_\infty$. Consequently, if $\mathcal{D}^+\xi = 0$, $\xi \in \mathcal{H}_\infty$, and similarly for $\mathcal{D}^-$. Hence the kernel and cokernel of $\mathcal{D}^+$ consist of elements of $\mathcal{H}_\infty$, and the index is independent of which 'Sobolev space' we use. The equality with $\text{Index}(\mathcal{D}^+(1 + \mathcal{D}^2)^{-1/2})$ now follows from the invertibility of $(1 + \mathcal{D}^2)^{-1/2}$. □

Example 5. In finite dimensions, i.e. $\dim \mathcal{H} < \infty$, we can take $\mathcal{A}$ to be finite dimensional, and so we are dealing with sums of matrix algebras. There is then really no constraint in the definition of spectral triple. If we have an even triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \gamma)$ which is finite dimensional in this sense, then

$$\text{Index}(\mathcal{D}_+ : \mathcal{H}_+ \to \mathcal{H}_-) = \dim \mathcal{H}_+ - \dim \mathcal{H}_-$$

by the rank nullity theorem.

The stability properties of the index are ultimately responsible for our ability to construct (co)homology theories using spectral triples and their relatives. This will be a theme for the rest of these notes.

2.4.2 Connes' metric for spectral triples

Here we mention one important geometric feature of spectral triples, the metric on the state space of the algebra. This is of some importance for the construction of spectral triples for particular algebras. The heuristic idea that the '$\mathcal{D}$' of a spectral triple is some sort of differentiation allows us to use metric ideas to construct '$\mathcal{D}$' so it is compatible with the notion of difference ratios coming from a given metric. We give examples below.

A state on a unital C^*-algebra is a linear functional $\phi : A \to \mathbb{C}$ with $\phi(1) = 1 = \|\phi\|$ and $\phi(a^*a) \geq 0$ for all $a \in A$. This is a convex space, and the extreme points (those states that are not convex combinations of other states) are called pure states. We denote the state space by $\mathcal{S}(A)$ and the pure states by $\mathcal{P}(A)$. The pure states of a commutative C^*-algebra, $C(X)$, correspond to point evaluations. So for $x \in X$ defining $\phi_x(f) = f(x)$, for all $f \in C(X)$ gives a pure state, and they are all of this form. Indeed the weak* topology on $\mathcal{P}(C(X))$ is the original topology on X, and $\mathcal{P}(C(X)) \simeq X$ (Gel'fand-Naimark Theorem).

We now show that spectral triples are 'noncommutative metric spaces'. We begin with the definition of the metric.

Lemma 2.23. *Suppose that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a spectral triple and that*

$$\{a \in \mathcal{A} \setminus 1 : \| [\mathcal{D}, a] \| \leq 1\} \tag{2.7}$$

is a norm bounded set in the Banach space $\overline{\mathcal{A}} \setminus \mathbb{C}1$. Then

$$d(\phi, \psi) = \sup_{a \in \mathcal{A}} \{|\phi(a) - \psi(a)| : \| [\mathcal{D}, a] \| \leq 1\}$$

defines a metric on $\mathcal{P}(\overline{\mathcal{A}})$, the pure state space of $\overline{\mathcal{A}}$.

Proof. The triangle inequality is a direct consequence of the definition. To see that $d(\phi, \psi) = 0$ implies $\phi = \psi$, suppose $\phi \neq \psi$. Then there is some $a \in \overline{\mathcal{A}}$ with $\phi(a) \neq \psi(a)$, and we can use the density of $\mathcal{A}$ in $\overline{\mathcal{A}}$ to find an element of $b \in \mathcal{A}$ such that $\phi(b) \neq \psi(b)$, and so $d(\phi, \psi) \neq 0$. The norm boundedness of the set in (2.7) gives the finiteness of the distance between any two pure states. □

In future, when we mention the metric associated to a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, we implicitly assume that the condition (2.7) is met. In particular, it means that no element of $\mathcal{A}$ except scalars commutes with $\mathcal{D}$. In fact in [RV], Lemma 2.23 was improved. The statement is

Proposition 2.24. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple such that $\mathcal{A}$ is unital, $\mathcal{A}\mathcal{H}$ is dense in $\mathcal{H}$, and $\mathcal{A}$ has separable norm closure A. Assume that that $[\mathcal{D}, a] = 0$ if and only if $a = \lambda 1$ for some $\lambda \in \mathbb{C}$. Then the formula*

$$d_{\mathcal{D}}(\phi, \psi) := \sup\{|\phi(a) - \psi(a)| : \|[\mathcal{D}, a]\| \leq 1\}$$

defines a metric on the state space of A.

The proof depends on showing that the distance between any two points is finite (though the distance need not be bounded).

The metric is actually defined on the whole state space $\mathcal{S}(A)$, but the metric on $\mathcal{S}(A)$ need not be determined by the restriction of the metric to the pure states, even for commutative examples, [Rie3]. Much more information about 'compact quantum metric spaces' is contained in [Rie1, Rie2, Rie3] and references therein. In particular, Rieffel proves that if the set in Equation 2.7 is in fact pre-compact in $\overline{\mathcal{A}} \setminus \mathbb{C}1$, then the metric induces the weak* topology on the state space.

Note that when $\mathcal{A}$ is commutative, so that $\mathcal{A}$ is an algebra of (at least continuous for the weak* topology) functions on $X = \mathcal{P}(\overline{\mathcal{A}})$, the metric topology on $\mathcal{P}(\overline{\mathcal{A}})$ is automatically finer than the weak* topology. In the case of a smooth Riemannian spin manifold, whose algebra of smooth functions is finitely generated by the (local) coordinate functions, not only do the topologies on the pure state space agree, so do the metrics, [C1, C2].

Lemma 2.25. *If $(C^\infty(M), L^2(E), \mathcal{D})$ is the spectral triple of any 'Dirac type operator' of the Clifford module E on a compact Riemannian spin manifold M, then*

$$d(\phi, \psi) = d_\gamma(\phi, \psi), \quad \forall \phi, \psi \in \mathcal{P}(C(M)),$$

where d_γ is the geodesic distance on X.

More generally, whenever $\mathcal{A}$ is commutative, we can take $\mathcal{A} \subseteq \mathrm{Lip}_d(\mathcal{P}(\mathcal{A}))$, the Lipschitz functions with respect to the metric topology, since

$$|a(\phi) - a(\psi)| := |\phi(a) - \psi(a)| \leq \| [\mathcal{D}, a] \| \, d(\phi, \psi)$$

for all $a \in \mathcal{A}$, $\phi, \psi \in \mathcal{P}(\mathcal{A})$.

Example 6. Here is a simple way to use metric ideas to build a spectral triple. Let $\mathcal{A} = \mathbb{C} \oplus \mathbb{C}$ be the continuous functions on two points. Let $\mathcal{A}$ act on the Hilbert space $\mathcal{H} = \mathbb{C}^2$ by multiplication. Let $0 \neq m \in \mathbb{R}$ and set $\mathcal{D} = \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}$ and the grading $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The index here is zero, but the distance between the two points is $\frac{1}{m}$. Check it yourself (**Exercise**).

If you want a nonzero index as well, let $(a, b) \in \mathcal{A}$ act on $\mathbb{C}^3$ by

$$\pi(a, b) \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} a\xi_1 \\ b\xi_2 \\ b\xi_3 \end{pmatrix}$$

and define $\mathcal{D} = \begin{pmatrix} 0_2 & \bar{m} \\ (\bar{m})^T & 0 \end{pmatrix}$ where now $\bar{m} = \begin{pmatrix} m \\ 0 \end{pmatrix}$ and $\gamma = 1 \oplus -1_2$.

Exercise What is the distance now?

You can have lots of fun building more complicated examples with different indices and different numbers of points and so on. However, as the number of points goes up, the expression for the distance (given a generic operator $\mathcal{D}$) becomes more and more complicated. It has been shown that for a particular class of examples of this form, polynomials of degree 5 and more arise, so the distance is generically not computable using arithmetic and the extraction of roots, [IKM].

Another level of complexity is added when we consider matrix algebras instead of copies of $\mathbb{C}$. This is because we can have much more complicated commutation relations. We refer to [IKM] for a fuller discussion of these examples, but recommend playing with some to get the feel for the problems.

These are *not* just trivial little toy models. Taking the product of the Dirac spectral triple of a manifold with certain spectral triples for sums of matrix algebras yields spectral triples with close relationships with particle physics. The reconstruction of the (classical) Lagrangian of the standard model of particle physics from such a procedure gave the subject an enormous boost. See [GV] for an introduction and a guide to some of the extensive literature on this subject.

For more information on these finite spectral triples, see [IKM, K, PS].

There are many other examples of spectral triples built with the intention of recovering or studying metric data (also dimension type data). Some interesting examples are contained in the recent papers of Erik Christensen and Cristina Ivan, as well as their coauthors. See [CIL, CI1, CI2, AC]. We present one more example here which we will look at again when we come to dimensions.

Example 7. This example is cute, and shows the kinds of pathologies which can crop up. Many thanks to Nigel Higson for relating it to me. Consider the Cantor 'middle thirds' set K. So start with the unit interval, and remove the (open) middle third. Then remove the open middle third of the two remaining subintervals etc. Observe that points in K come in pairs, e_-, e_+ where e_- is the left endpoint of a gap and e_+ is the right endpoint of a gap. Every point except $0, 1$ is one (and only one) of these two types, and we take $0, 1$ as a pair.

Let $\mathcal{H} = l^2(\textit{end points})$ and $\mathcal{A}$ be the locally constant functions on K. Recall that a function $f : K \to \mathbb{C}$ is locally constant if for all $x \in K$ there is a neighbourhood U of x such that f is constant on U. Define $\mathcal{D} : \mathcal{A} \cap \mathcal{H} \to \mathcal{H}$ by

$$(\mathcal{D}f)(e_+) = \frac{-f(e_-)}{e_+ - e_-}, \quad (\mathcal{D}f)(e_-) = \frac{f(e_+)}{e_+ - e_-}.$$

The closure of this densely defined operator is self-adjoint (Exercise). Also

$$[\mathcal{D}, f]g(e_+) = \frac{f(e_+) - f(e_-)}{e_+ - e_-} g(e_-), \quad [\mathcal{D}, f]g(e_-) = \frac{f(e_+) - f(e_-)}{e_+ - e_-} g(e_+).$$

For f locally constant the commutator $[\mathcal{D}, f]$ defines a bounded operator.

Let δ_{e_+} be the function which is one on e_+ and zero elsewhere, and similarly for δ_{e_-}. Now these are not locally constant functions, but are in the domain of (the closure of) $\mathcal{D}$. We observe that $\text{span}\{\delta_{e_+}, \delta_{e_-}\}$ is invariant under $\mathcal{D}$ since

$$\mathcal{D}\delta_{e_+} = \delta_{e_-}, \quad \mathcal{D}\delta_{e_-} = -\delta_{e_+}.$$

Indeed in the basis given by $\delta_{e_+}, \delta_{e_-}$, $\mathcal{D} = \frac{1}{e_+ - e_-} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Hence $\mathcal{D}$ has eigenvalues $\pm 1/(e_+ - e_-) = \pm 3^n$ if the points appear in the n-th stage of the construction, and their multiplicity is 2^{n-1}. Thus $(1 + \mathcal{D}^2)^{-1}$ is certainly compact and invertible.

It is now an **Exercise** to show that Connes metric is precisely the usual metric on the Cantor set. We will return to this example when we discuss summability.

Chapter 3

More spectral triples from manifolds

Our aim in this Chapter is to produce more examples of spectral triples from manifolds. To do this we will use a little more differential geometry, as well as the Clifford algebra formalism.

The constructions we use turn out to have a fairly deep interpretation in KK-theory, and yield an intricate notion of Poincaré Duality in K-theory. This would be the topic of a second (or third...) course.

3.1 The signature operator

In dimensions $n = 4k$ there is another grading on the space $\Gamma(\Lambda^* T^* M)$ that allows us to define a new spectral triple. In these dimensions, the complex volume form $\omega_{\mathbb{C}}$ is given by $(-1)^k \omega = (-1)^k e_1 \cdots e_{4k}$ in terms of a local orthonormal basis. Consequently, the Clifford action by $\omega_{\mathbb{C}}$ maps the space of real sections of $\Lambda^* M$ into itself. Moreover, for $\varphi \in \Lambda^p M$ we have

$$\omega_{\mathbb{C}} \cdot \varphi = (-1)^{k+p(p-1)/2} * \varphi,$$

where $*$ is the Hodge star operator. Since (in even dimensions) $d^* = - * d *$, we see that $d + d^*$ anticommutes with the action of $\omega_{\mathbb{C}}$, and we get a new grading. We already know that $d + d^*$ has compact resolvent, so when $\dim M = n = 4k$,

$$(C^\infty(M), L^2(\Lambda^* M, g), d + d^*, \omega_{\mathbb{C}})$$

is an even spectral triple.

What is the index? Well, the identification up to sign of $\omega_{\mathbb{C}} \cdot$ and $*$ gives us isomorphisms

$$\omega_{\mathbb{C}} : \mathbf{H}^p \to \mathbf{H}^{4k-p}$$

for each $p = 0, 1, \ldots, 2k$.

Proof. We know from the Hodge decomposition theorem that $\varphi \in \mathbf{H}^p$ if and only if $d\varphi = d^*\varphi = 0$. So let $\varphi \in \mathbf{H}^p$. Then $d\omega_{\mathbb{C}}\varphi = \pm\omega_{\mathbb{C}} d^*\varphi = 0$ and similarly, $d^*\omega_{\mathbb{C}}\varphi = \pm\omega_{\mathbb{C}} d\varphi = 0$. Since $\omega_{\mathbb{C}}^2 = 1$, we are done. □

So for $p = 0, \dots, 2k-1$ the space $\mathbf{H}(p) = \mathbf{H}^p \oplus \mathbf{H}^{4k-p}$ has a decomposition

$$\mathbf{H}(p) = \mathbf{H}^+(p) \oplus \mathbf{H}^-(p) = \frac{(1+\omega_{\mathbb{C}})}{2}\mathbf{H}(p) \oplus \frac{(1-\omega_{\mathbb{C}})}{2}\mathbf{H}(p).$$

Observe that the subspaces $\frac{1}{2}(1 \pm \omega_{\mathbb{C}})\mathbf{H}(p)$ have the same dimension.

Proof. If $\{\varphi_1, \dots, \varphi_m\}$ is a basis of $\mathbf{H}^p$, the subspace $\mathbf{H}^+(p)$ has basis $\{\varphi_1 + \omega_{\mathbb{C}}\varphi_1, \dots, \varphi_m + \omega_{\mathbb{C}}\varphi_m\}$. Likewise $\mathbf{H}^-(p)$ has basis $\{\varphi_1 - \omega_{\mathbb{C}}\varphi_1, \dots, \varphi_m - \omega_{\mathbb{C}}\varphi_m\}$. $\square$

This allows us to compute

$$\ker \mathcal{D}^{\pm} = \mathbf{H}^{\pm} := \mathbf{H}^{\pm}(0) \oplus \mathbf{H}^{\pm}(1) \oplus \cdots \oplus \mathbf{H}^{\pm}(2k-1) \oplus (\mathbf{H}^{2k})^{\pm},$$

where $(\mathbf{H}^{2k})^{\pm} = (1 \pm \omega_{\mathbb{C}})\mathbf{H}^{2k}$. Since the index is $\dim\ker\mathcal{D}^+ - \dim\ker\mathcal{D}^-$, we have

$$\mathrm{Index}(d + d^*) = \dim(\mathbf{H}^{2k})^+ - \dim(\mathbf{H}^{2k})^-.$$

Observe that on $\Lambda^{2k}M$, $\omega_{\mathbb{C}}$ and the Hodge star operator coincide. We define a new inner product on $\mathbf{H}^{2k}$ by

$$(\phi|\psi)^{new} := \int_M \phi \wedge \psi.$$

Then using $* = \omega_{\mathbb{C}}$ and $\int_M \phi \wedge *\phi = \|\phi\|^2$, we see that the signature of this inner product (the number of positive eigenvalues minus the number of negative eigenvalues) is precisely the difference in dimensions of the ± 1 eigenspaces of the $*$ operator. So letting signature(M) denote the signature of this inner product,

$$\mathrm{Index}((d + d^*)^+, \omega_{\mathbb{C}}) = \mathrm{signature}(M).$$

Remarkably, this is a homotopy invariant of the manifold.

3.2 Connections and twistings

We can make much of our discussion more streamlined by taking fuller advantage of Clifford algebras.

Recall that if W is a left module over a $*$-algebra $\mathcal{A}$, an hermitian form is a map $(\cdot|\cdot) : W \times W \to \mathcal{A}$ such that for all $v, w, y \in W$ and $a \in \mathcal{A}$

$$(v + w|y) = (v|y) + (w|y), \quad (av|y) = a(v|y), \quad (v|y) = (y|v)^*.$$

If $\mathcal{A}$ is a pre-C^*-algebra, we can also ask for $(v|v) \geq 0$ in the sense of the C^*-closure of $\mathcal{A}$, and that $(v|v) = 0 \Rightarrow v = 0$. We assume all our hermitian forms satisfy these properties.

For a vector bundle $E \to M$, we know that $\Gamma(E)$ is a module over $C^\infty(M)$ via $(f\sigma)(x) = f(x)\sigma(x)$ for all $f \in C^\infty(M)$, $\sigma \in \Gamma(E)$ and $x \in M$. An hermitian form is then a collection of positive definite inner products $(\cdot|\cdot)_x$ on E_x such that for all smooth sections $\sigma, \rho \in \Gamma(E)$, the function $x \to (\sigma(x)|\rho(x))_x$ is smooth. All complex vector bundles have such a smooth inner product.

Now let $E \to M$ be a smooth vector bundle, and let $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$ be a connection. So ∇ is $\mathbb{C}$-linear and for all $f \in C^\infty(M)$ and $\sigma \in \Gamma(E)$

$$\nabla(f\sigma) = df \otimes \sigma + f\nabla\sigma.$$

We can extend ∇ to a map $\nabla : \Gamma(\Lambda^*M \otimes E) \to \Gamma(\Lambda^*M \otimes E)$ by defining for $\omega \in \Gamma(\Lambda^k M)$ and $\sigma \in \Gamma(E)$

$$\nabla(\omega \otimes \sigma) = d\omega \otimes \sigma + (-1)^k \omega \otimes \nabla\sigma.$$

One of the most important observations is that ∇^2 is linear over $C^\infty(M)$.

Proof. Let $f \in C^\infty(M)$ and $\sigma \in \Gamma(E)$. Then

$$\nabla^2(f\sigma) = \nabla(df \otimes \sigma + f\nabla\sigma) = d^2 f \otimes \sigma - df \otimes \nabla\sigma + df \otimes \nabla\sigma + f\nabla^2\sigma = f\nabla^2\sigma.$$

□

Thus ∇^2 is a two-form with values in the endomorphisms of E (locally a matrix of two-forms). It is called the curvature of E.

A connection ∇ on a vector bundle $E \to M$ with hermitian form $(\cdot|\cdot)$ is said to be compatible with $(\cdot|\cdot)$ if for all $\sigma, \rho \in \Gamma(E)$

$$d(\sigma|\rho) = (\nabla\sigma|\rho) + (\sigma|\nabla\rho),$$

where to interpret the right hand side we write in local coordinates $\nabla\sigma = \sum_i dx^i \otimes \sigma_i$ and $\nabla\rho = \sum_j dx^j \otimes \rho_j$ and then

$$(\nabla\sigma|\rho) + (\sigma|\nabla\rho) := \sum_i dx^i(\sigma_i|\rho) + \sum_j dx^j(\sigma|\rho_j).$$

Example 8. The Levi-Civita connection on TM or T^*M is compatible with the Riemannian metric. The curvature of the Levi-Civita connection is, by definition, the curvature of the manifold.

Lemma 3.1 (see [LM]). *Let M be a compact oriented manifold, and let c denote the usual left action of $\mathbb{C}liff(M)$ on Λ^*M. Let ∇ denote the Levi-Civita connection on T^*M. then*

$$d + d^* = c \circ \nabla.$$

Thus the Hodge de-Rham and Signature operators are both given by composing the Levi-Civita connection with the Clifford action. This is a very general recipe, and allows us to construct twisted versions of these operators.

If $E, F \to M$ are both vector bundles, with connections ∇^E, ∇^F respectively, we can define a connection $\nabla^{E,F}$ on the tensor product $E \otimes F$ by defining for all $\sigma \in \Gamma(E)$ and $\rho \in \Gamma(F)$

$$\nabla^{E,F}(\sigma \otimes \rho) = (\nabla^E\sigma) \otimes \rho + \sigma \otimes (\nabla^F\rho).$$

If E is a $\mathbb{C}liff(M)$ module, then so is $E \otimes F$ by letting $\mathbb{C}liff(M)$ act only on E. Thus we can form the operator

$$c \circ \nabla^{E,F} : \Gamma(E \otimes F) \to \Gamma(E \otimes F).$$

Choose an hermitian structure on E so that for any one form φ

$$(c(\varphi)\rho|\sigma) = -(\rho|c(\bar{\varphi})\sigma), \quad \rho, \sigma \in E,$$

where c denotes the Clifford action. Such an inner product always exists. This ensures that $c \circ \nabla^{E,F}$ is (formally) self-adjoint.

Applying this recipe to $d + d^*$ allows us to 'twist' $d + d^*$ by any vector bundle

$$d + d^* \otimes_\nabla Id_E := c \circ \nabla^{T^*M,E}.$$

By choosing a connection compatible with a product Hermitian structure, and using the integral to define a scalar inner product, we can construct a new spectral triple

$$(C^\infty(M), L^2(\Lambda^*M \otimes E, g \otimes (\cdot|\cdot)), d + d^* \otimes_\nabla Id_E).$$

Remark We have made use of the commutativity of the algebra at a few points in the above discussion. For example, we have identified the right and left actions of functions on sections by multiplication. This allows us to use $\Gamma(E \otimes F) = \Gamma(E) \otimes_{C^\infty(M)} \Gamma(F)$.

Many of these tricks are unavailable in the noncommutative case. If we have a spectral triple $(\mathcal{A} \otimes \mathcal{A}^{op}, \mathcal{H}, \mathcal{D})$, where $\mathcal{A}^{op}$ is the opposite algebra, then we can twist by finite projective modules $p\mathcal{A}^N$ or equivalently $p\mathcal{A}^{op,N}$, and be left with a spectral triple for one copy of $\mathcal{A}$. This point of view underlies Poincaré duality in K-theory.

3.3 The spin^c condition and the Dirac operator

Building differential operators from connections and Clifford actions yields operators which depend on the Riemannian metric and the Clifford module. The index of such an operator is an invariant of the manifold and the Clifford module, [BGV, Thm 3.51]. Since for both the Hodge-de Rham and signature operator the Clifford module depends only on the manifold, we see that the Euler chacteristic and the signature are invariants of the manifold.

The question is, what other kinds of operators can one build in this way? For spin and spin^c manifolds, there is an interesting answer.

The spin^c condition was originally formulated in differential geometry language, involving double covers of principal SO bundles on a manifold. This brings in the spin groups and their representations. This will take us too far afield, so we will take a different approach more suitable for noncommutative geometry.

The definition of spin^c has been shown by Plymen, [P], to be equivalent to the following straightforward characterisation in terms of Clifford algebras.

Definition 3.2 ([P])**.** *Let (M, g) be an oriented Riemannian manifold. Then we say that (M, g) is spin^c if there exists a complex vector bundle $S \to M$ such that for all $x \in M$ the vector space S_x is an irreducible representation space for $\mathbb{C}liff_x(M, g)$.*

A spin^c structure on a Riemannian manifold (M, g) is then the choice of an orientation and irreducible representation bundle S of $\mathbb{C}liff(M, g)$. The bundle S is called a (complex) spinor bundle.

When M has at least one spin^c structure S, we can build a new operator called the Dirac operator. Choose an Hermitian form $(\cdot|\cdot)$, let $\nabla : \Gamma(S) \to \Gamma(T^*M \otimes S)$ be any connection compatible with $(\cdot|\cdot)$, and compose the connection with the Clifford action:

$$\Gamma(S) \xrightarrow{\nabla} \Gamma(T^*M \otimes S) \xrightarrow{c} \Gamma(S).$$

The resulting operator $\mathcal{D} = c \circ \nabla : \Gamma(S) \to \Gamma(S)$ is the Dirac operator on the (complex) spinor bundle S.

Using the more geometric definitions in terms of principal bundles, one can obtain a more canonical Dirac operator by taking ∇ to be a 'lift' of the Levi-Civita connection to S.

By looking at connections on a $\mathbb{C}liff(M)$ module, we see that we have a general construction of a Dirac operator on any such module. The following Lemma tells us what these modules look like.

Lemma 3.3 (see [BGV])**.** *If M is a spin^c manifold, then every $\mathbb{C}liff(M)$ module is of the form $S \otimes W$ where S is an irreducible $\mathbb{C}liff(M)$ module and W is a complex vector bundle.*

Thus on a spin^c manifold, we can describe every 'Dirac type' operator as a twisted version of the Dirac operator of an irreducible Clifford module. In fact, using Poincaré duality in K-theory, one can show that up to homotopy and change of the order of the operator, every elliptic operator on a compact spin^c manifold is a twisted Dirac operator. See [HR, R2].

One very important difference between the Dirac operator of a spin^c structure and the Hodge-de Rham operator must be mentioned. Recall the complex volume form defined in terms of a local orthonormal basis of the cotangent bundle by $\omega_{\mathbb{C}} = i^{[(n+1)/2]} e_1 \cdot e_2 \cdot \cdots \cdot e_n$. This element of the Clifford algebra is globally parallel, meaning $\nabla \omega_{\mathbb{C}} = 0$, where ∇ is the Levi-Civita connection. Since for any differential one form φ we have

$$\varphi \omega_{\mathbb{C}} = (-1)^{n-1} \omega_{\mathbb{C}} \varphi$$

we have the following two situations.

When n is odd, $\omega_{\mathbb{C}}$ is central with eigenvalues ± 1. Since the Clifford algebra is (pointwise) $M_{2^{(n-1)/2}}(\mathbb{C}) \oplus M_{2^{(n-1)/2}}(\mathbb{C})$, we can take $\omega_{\mathbb{C}} = 1 \oplus -1$. An irreducible representation of $\mathbb{C}liff(M)$ then corresponds (pointwise) to a representation of one of the two matrix subalgebras. Without loss of generality we choose the representation with $\omega_{\mathbb{C}} = 1$. In this case the spectral triple $(C^\infty(M), L^2(S), \mathcal{D})$ is ungraded, or odd.

When n is even, the ± 1 eigenspaces of $\omega_{\mathbb{C}}$ provide a global splitting of $S = S^+ \oplus S^-$. The Clifford action of a one form maps S^+ to S^- and vice versa. Hence

$$\omega_{\mathbb{C}} \mathcal{D} = -\mathcal{D} \omega_{\mathbb{C}}.$$

As the Clifford algebra is (pointwise) a single matrix algebra, we get a representation of the whole Clifford algebra. The resulting spectral triple $(C^\infty(M), L^2(S), \mathcal{D})$ is graded by the action of $\omega_{\mathbb{C}}$, and we get an even spectral triple.

Thus the Dirac operator of a spin^c structure gives an even spectral triple if and only if the dimension of M is even. The same remains true if we twist the Dirac operator by any vector bundle.

One can define a spin structure in terms of representations of real Clifford algebras. This is not quite a straightforward generalisation, but does go through: see [LM, GVF, P]. Thus one can talk about the Dirac operator of a spin structure also. In many ways this is easier, and certainly is so from the differential geometry point of view, see [LM, Appendix D].

3.4 The Atiyah-Singer index theorem

The Hodge-de Rham operator has index equal to the Euler characteristic of the manifold. In dimension 2, the Gauss-Bonnet Theorem asserts that

$$\chi(M) = \frac{1}{2\pi}\int_M r\, dvol,$$

where r is the scalar curvature of M. This is a remarkable formula, because it allows us to compute a topological quantity, $\chi(M)$, using geometric quantities. More blatantly, it says that by doing explicitly computable calculus operations, we can compute this topological invariant. The answer does not depend on which coordinates or metric we choose to compute with.

The Atiyah-Singer Index Theorem generalises this theorem to any elliptic operator. Specifically, it says:

- give me a first order elliptic operator $D : \Gamma(E) \to \Gamma(F)$ between sections of (complex) vector bundles E, $F \to M$;

- I will give you a sum of even differential forms $\omega(D) = \omega_0(D) + \cdots + \omega_{2[n/2]}(D)$ so that

$$\text{Index}(D) = \int_M \omega_{2[n/2]}(D).$$

In particular, if n is odd the index is zero.

However, more is true. If you take a vector bundle W and twist everything to get $D \otimes_\nabla Id_W$ then

$$\text{Index}(D \otimes_\nabla Id_W) = \int_M \omega(D) \wedge Ch(W).$$

Here $Ch(W)$ is the Chern character of W defined by $Ch(W) = \text{Trace}(e^{-\nabla^2})$ where ∇ is any connection on W. We will give another description of the Chern character of vector bundles later.

Thus the Atiyah-Singer Index Theorem allows you to not only compute the index of D, but all twisted versions of D also. This means that D is a machine for turning vector bundles into integers via

$$W \mapsto \text{Index}(D \otimes_\nabla Id_W) \in \mathbb{Z}.$$

This is actually 'the same' in odd dimensions, the difference being that the Chern character of a unitary (see Appendix B) has only odd degree differential form components. Thus $\omega(D) \wedge Ch(u)$ is an odd form, and so there can be forms of degree $\dim M$ to integrate. Also, it is not so straightforward to write down a differential operator whose index we are computing (it can be done) and we should think of the odd case as computing $\text{Index}(PuP)$ where $P = \chi_{[0,\infty)}(\mathcal{D})$ is the positive spectral projection of $\mathcal{D}$.

Example 9. Hodge-de Rham. Here the sum of differential forms ω is given by $(2\pi)^{-n/2}Pf(-R)$, where Pf is short for the Pfaffian of an antisymmetric matrix, and R is the curvature of the Levi-Civita connection. The Pfaffian satisfies $Pf(A)^2 = \det(A)$, and changes sign if the orientation changes.

In the case where $\dim M = 2$, $Pf(-R) = r$, the scalar curvature, and so

$$\mathrm{Index}((d + d^*)^+) = \chi(M) = \frac{1}{2\pi}\int_M r dvol,$$

and we recover the classical Gauss-Bonnet theorem.

Example 10. For the signature operator,

$$\mathrm{Index}((d + d^* \otimes_\nabla Id_E)^+) = (\pi i)^{-n/2}\int_M L(M) \wedge Ch(E),$$

where the L-genus is $L = 1 + \frac{1}{24}\mathrm{Tr}(R^2) + \cdots$.

Example 11. For the spin Dirac operator,

$$\mathrm{Index}((\mathcal{D} \otimes_\nabla Id_E)^+) = (2\pi i)^{-n/2}\int_M \hat{A}(M) \wedge Ch(E),$$

where $\hat{A}$ is called the 'A-roof' class. So

$$\mathrm{Index}(\mathcal{D}^+) = (2\pi i)^{-n/2}\int_M \hat{A}(M).$$

Since $\hat{A}(M)$ is defined in terms of the curvature, the index is independent of the spin structure. The right hand side is always a rational number, and if it is not an integer, the manifold has no spin structure.

These examples, and discussion of the Atiyah-Singer index theorem, are presented in [BGV, G, LM].

A natural question is whether the schematic

Index equals integral of differential forms

has any sensible generalisation for noncommutative algebras. In a very real sense, cyclic homology is the generalisation of de Rham cohomology, and obtaining formulae for the index in terms of cyclic homology and cohomology is analogous to integrating differential forms to compute the index. We will take this up later.

3.5 The noncommutative torus

One of the nicest and most thoroughly studied spectral triples is defined for the irrational rotation algebra (and its higher dimensional relatives). The spectral triple defined below satisfies every proposed condition intended to characterise what we mean by a noncommutative manifold, [C3]. So whenever everyone agrees on what a noncommutative manifold is, the noncommutative torus will be an example.

The noncommutative torus is the universal unital C^*-algebra A_θ generated by two unitaries subject to the commutation relations

$$UV = e^{-2\pi i\theta}VU, \quad \theta \in [0,1). \tag{3.1}$$

For $\theta = 0$ this is clearly the algebra of continuous functions on the torus. For θ rational, we obtain an algebra Morita equivalent to the functions on the torus. We will be interested in the case where θ is irrational. In this case, A_θ is simple.

There are two other descriptions of A_θ. The first is as the C^*-algebra associated to the Kronecker foliation of the (ordinary) torus given by the differential equation

$$dy = \theta dx. \tag{3.2}$$

Spectral triples can be constructed for more general foliations also, see [Ko].

The other description is as a crossed product. For this description we have

$$A_\theta = C(S^1) \times_{R_\theta} \mathbb{Z}, \tag{3.3}$$

where $U = z \in C(S^1)$ is the generator of functions on the circle, and V implements the rotation by $2\pi\theta$:

$$R_\theta(U) = VUV^* = e^{2\pi i\theta}U. \tag{3.4}$$

There is to my knowledge no general construction of spectral triples for crossed products $A \rtimes \Gamma$ given a discrete group Γ and spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. Some special cases can be found in the literature.

In order to build a spectral triple encoding geometry on the noncommutative torus, we need a smooth algebra, an unbounded operator and a Hilbert space. We begin with the algebra. Let

$$\mathcal{A}_\theta = \{\sum_{n,m\in\mathbb{Z}} c_{nm}U^nV^m : |c_{nm}|(|n| + |m|)^q \text{ is a bounded double sequence for all } q \in \mathbb{N}\}. \tag{3.5}$$

Fourier theory on the ordinary torus suggests viewing this algebra as the smooth functions on the noncommutative torus. It is not much work to see that $\mathcal{A}_\theta$ is indeed a Fréchet pre-C^*-algebra (see Section 5.1).

Next we require a Hilbert space. Recall that for θ irrational, A_θ has a unique faithful normalised trace ϕ given (on polynomials in the generators) by

$$\phi(a) = \phi(\sum c_{ij}U^iV^j) = c_{00}. \tag{3.6}$$

If we set $\langle a, b\rangle = \phi(b^*a)$, then $\langle\cdot,\cdot\rangle$ is an inner product and this makes A_θ a pre-Hilbert space. Completing with respect to the topology given by the inner product gives us a Hilbert space called $L^2(A_\theta, \phi)$. The algebra A_θ acts on $L^2(A_\theta, \phi)$ in the obvious way as multiplication operators. We set

$$\mathcal{H} = L^2(A_\theta, \phi) \oplus L^2(A_\theta, \phi). \tag{3.7}$$

This doubling up of the Hilbert space is motivated by the dimension of spinor bundles on the ordinary torus, or equivalently, the dimension of irreducible representations of $\mathbb{C}liff(\mathbb{C}^2)$. Thus, loosely speaking, we are aiming to build a Dirac triple rather than a Hodge-de Rham triple.

In order to specify a Dirac operator for our triple, we need to look at how geometric data are encoded for classical tori. The problem is that any quadrilateral with opposite sides identified gives rise to a *geometric* object which is homeomorphic to the torus. To specify the extra geometric content given by our original quadrilateral, we embed it in the first quadrant of the complex plane with one vertex at the origin and another at $1 \in \mathbb{R}$ (we could scale the geometry up by putting one corner at $r \in \mathbb{R}$, but this is more or less irrelevant). The resulting geometry is then specified by the ratio of the edge lengths as complex numbers, or with our description, a single complex number τ which is the coordinate of the other independent vertex. In particular, $Im(\tau) > 0$. The usual 'square' torus corresponds to the choice $\tau = i$.

With this in mind we define two derivations on A_θ by

$$\delta_1(U) = 2\pi i U \qquad \delta_1(V) = 0 \tag{3.8}$$

$$\delta_2(U) = 0 \qquad \delta_2(V) = 2\pi i V. \tag{3.9}$$

We then find (using $UU^* = 1$ etc and the Leibnitz rule) that $\delta_1(U^*) = -2\pi i U^*$, $\delta_2(V^*) = -2\pi i V^*$, $\delta(U^n) = n 2\pi i U^n$ and so on. These rules correspond to the derivatives of exponentials generating the functions on a torus. Using these derivations and a choice of τ with $Im(\tau) > 0$ we define

$$\mathcal{D} = \begin{pmatrix} 0 & \delta_1 + \tau\delta_2 \\ -\delta_1 - \bar{\tau}\delta_2 & 0 \end{pmatrix}. \tag{3.10}$$

Lastly, we set

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.11}$$

Observe that we have the following heuristic. Since

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \tau \\ -\bar{\tau} & 0 \end{pmatrix} + \begin{pmatrix} 0 & \tau \\ -\bar{\tau} & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -2Re(\tau) & 0 \\ 0 & -2Re(\tau) \end{pmatrix},$$

it looks like we are working with the Clifford algebra of the inner product

$$g = \begin{pmatrix} 1 & Re(\tau) \\ Re(\tau) & |\tau|^2 \end{pmatrix}.$$

Again, if $\tau = i$, we are reduced to the usual Euclidean inner product.

Hard question Can we encode a nonconstant metric using this heuristic?

I claim $(\mathcal{A}_\theta, \mathcal{H}, \mathcal{D})$ defines an even spectral triple with grading γ for each such choice of τ. First we must show that for all $a \in \mathcal{A}_\theta$ we have $[\mathcal{D}, a]$ bounded. So let $a = \sum c_{nm} U^n V^m \in \mathcal{A}_\theta$. Then

$$\begin{aligned} \mathcal{D}a - a\mathcal{D} &= \begin{pmatrix} 0 & \delta_1 + \tau\delta_2 \\ -\delta_1 - \bar{\tau}\delta_2 & 0 \end{pmatrix}\begin{pmatrix} \sum c_{nm}U^nV^m & 0 \\ 0 & \sum c_{nm}U^nV^m \end{pmatrix} \\ &- \begin{pmatrix} \sum c_{nm}U^nV^m & 0 \\ 0 & \sum c_{nm}U^nV^m \end{pmatrix}\begin{pmatrix} 0 & \delta_1 + \tau\delta_2 \\ -\delta_1 - \bar{\tau}\delta_2 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 2\pi i \sum c_{nm}U^nV^m(n + m\tau) \\ -2\pi i \sum c_{nm}U^nV^m(n + m\bar{\tau}) & 0 \end{pmatrix}, \end{aligned} \tag{3.12}$$

and as $|c_{nm}|$ is 'Schwartz class', this converges in norm and so is bounded.

Next we must show that $\mathcal{D}$ has compact resolvent. As this is equivalent to $\mathcal{D}$ having only eigenvalues of finite multiplicity (which must go to infinity so that $\mathcal{D}$ is unbounded) we will prove this instead. We begin by looking at $\mathcal{D}^2$,

$$\mathcal{D}^2 = \begin{pmatrix} -\delta_1^2 - |\tau|^2\delta_2^2 - \bar{\tau}\delta_1\delta_2 - \tau\delta_2\delta_1 & 0 \\ 0 & -\delta_1^2 - |\tau|^2\delta_2^2 - \tau\delta_1\delta_2 - \bar{\tau}\delta_2\delta_1 \end{pmatrix}. \tag{3.13}$$

Applying this to the monomial $U^nV^m \oplus U^nV^m$ gives

$$\begin{aligned} D^2 \begin{pmatrix} U^nV^m \\ U^nV^m \end{pmatrix} &= (2\pi)^2(n^2 + |\tau|^2m^2 + nm(\tau + \bar{\tau})) \begin{pmatrix} U^nV^m \\ U^nV^m \end{pmatrix} \\ &= (2\pi)^2|n + \tau m|^2 \begin{pmatrix} U^nV^m \\ U^nV^m \end{pmatrix}. \end{aligned} \tag{3.14}$$

This shows that all of these monomials are eigenvectors of $\mathcal{D}^2$. Note that

$$\begin{aligned} \phi(V^{-l}U^{-k}U^nV^m) &= \phi(V^{-l}U^{n-k}V^m) \\ &= \phi(e^{-2\pi il\theta(n-k)}U^{n-k}V^{m-l}) \\ &= \delta_{n,k}\delta_{m,l}, \end{aligned} \tag{3.15}$$

so that the monomials U^nV^m form an orthonormal basis of $L^2(A_\theta, \phi)$ (they clearly span). As $\mathcal{D}^2$ preserves the splitting of $\mathcal{H}$, we see that these are all the eigenvalues of $\mathcal{D}^2$ and that they give the whole spectrum of $\mathcal{D}^2$. Also note in passing that

$$\ker \mathcal{D}^2 = span_{\mathbb{C}}\{1\} \oplus span_{\mathbb{C}}\{1\} = \mathbb{C} \oplus \mathbb{C}. \tag{3.16}$$

Our results so far are actually enough to conclude, but let us make the eigenvalues and eigenvectors of $\mathcal{D}$ explicit.

The eigenvalues of $\mathcal{D}$ are given by the square roots of the eigenvalues of $\mathcal{D}^2$, and so are

$$\pm 2\pi|n + \tau m| \quad n, m \in \mathbb{Z}. \tag{3.17}$$

The corresponding eigenvectors are

$$+ve \quad \begin{pmatrix} \frac{i(n+\tau m)}{|n+\tau m|}U^nV^m \\ U^nV^m \end{pmatrix} \quad -ve \quad \begin{pmatrix} \frac{i(n+\tau m)}{|n+\tau m|}U^nV^m \\ -U^nV^m \end{pmatrix}. \tag{3.18}$$

The multiplicity of these eigenvalues depends on the value of τ, but is always finite. Thus $\mathcal{D}$ has compact resolvent, and for any choice of τ with $Im(\tau) > 0$ we have an even spectral triple.

Research project Describe the state space of A_θ and Connes' metric on this state space.

3.6 Isospectral deformations

Using the noncommutative torus we can construct other 'noncommutative manifolds'. Let (M, g) be a compact Riemannian manifold with an isometric action of $\mathbb{T}^2$ (we can do this with higher dimensional noncommutative tori also).

Then we define $C^\infty(M_\theta)$ to be the fixed point for the diagonal action of $\mathbb{T}^2$ on $C^\infty(M) \otimes A_\theta$. This is like gluing in a noncommutative torus in place of each torus orbit in M.

The same kind of procedure allows one to take $(C^\infty(M), \mathcal{H}, \mathcal{D})$ and produce $(C^\infty(M_\theta), \mathcal{H}_\theta, \mathcal{D})$, where $\mathcal{D}$ is essentially the same operator in both triples, and certainly has the same spectrum. The interested reader can look at the papers by Connes and Dubois-Violette, [CD].

Chapter 4

K-theory, K-homology and the index pairing

4.1 K-theory

This section is the briefest of overviews of K-theory for C^*-algebras. If the discussion here is unfamiliar, try [WO, RLL, HR]. We will take much of the discussion here from [HR].

4.1.1 K_0

Definition 4.1. *Given a unital C^*-algebra A, we denote by $K_0(A)$ the abelian group with one generator $[p]$ for each projection p in each matrix algebra $M_n(A)$, $n = 1, 2, \dots$ and the following relations:*

a) if p, $q \in M_n(A)$ and p, q are joined by a norm continuous path of projections in $M_n(A)$ then $[p] = [q]$;

b) $[0] = 0$ for any square matrix of zeroes;

c) $[p] + [q] = [p \oplus q]$ for any $[p]$, $[q]$.

In a), we say that p and q are homotopic. If $p \in M_n(A)$ we say that p is a projection over A.

Every element of $K_0(A)$ can be written as a formal difference $[p] - [q]$, and two elements $[p] - [q]$ and $[p'] - [q']$ are equal if and only if there is a projection r such that

$$p \oplus q' \oplus r \text{ is homotpic to } p' \oplus q \oplus r.$$

Exercise Prove this.

The group K_0 is a covariant functor from C^*-algebras to abelian groups. If $\phi : A \to B$ is a $*$-homomorphism, then applying ϕ element by element to the matrix $p \in M_n(A)$ gives a projection $\phi(p) \in M_n(B)$. This yields a group homomorphism $\phi_* : K_0(A) \to K_0(B)$.

Exercise Show that two projections in $M_n(\mathbb{C})$ are homotopic if and only if they have the same rank, and that $[p] \to \mathrm{Rank}(p)$ is an isomorphism from $K_0(\mathbb{C})$ to $\mathbb{Z}$.

Example 12. (**Important**) If $A = C(X)$, where X is a compact Hausdorff space, we find that $K_0(A) = K^0(X)$, where $K^0(X)$ is the topological K-theory defined by vector bundles.

In essence this is because if $E \to X$ is a complex vector bundle, there is a projection $p \in M_N(C(X))$ for some N such that $\Gamma(X, E) \cong pC(X)^N$ as a $C(X)$ module. Similarly, any $C(X)$ module of the form $pC(X)^N$ is the sections of a vector bundle. This is the Serre-Swan Theorem, [S].

The exchange between projections and vector bundles is one of the many important instances of exchanging topological information for algebraic information, with the Gel'fand-Naimark Theorem (exchanging abelian C^*-algebras and Hausdorff spaces) being one of the main motivations of noncommutative geometry.

4.1.2 K_1

Definition 4.2. *Given a unital C^*-algebra A, we denote by $K_1(A)$ the abelian group with one generator $[u]$ for each unitary u in each matrix algebra $M_n(A)$, $n = 1, 2, \ldots$ and the following relations:*

a) if u, $v \in M_n(A)$ and u, v are joined by a norm continuous path of unitaries in $M_n(A)$ then $[u] = [v]$;

b) $[1] = 0$ for any square identity matrix;

c) $[u] + [v] = [u \oplus v]$ for any $[u]$, $[v]$.

Exercise Let $\sim$ denote the relation of path-connectedness through unitaries. Let u, $v \in A$ be unitary. Prove that in $M_2(A)$ we have

$$u \oplus 1 \sim 1 \oplus u, \quad u \oplus v \sim uv \oplus 1 \sim vu \oplus 1, \quad u \oplus u^* \sim 1 \oplus 1.$$

Hint: consider the rotation matrix

$$R_t = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}.$$

From the exercise we see that $[u] + [u^*] = [u \oplus u^*] = [1] = 0$, and so $-[u] = [u^*]$.

Exercise Show that $K_1(\mathbb{C}) = 0$.

This section on K-theory could be made arbitrarily long, but it is not the main focus of these notes, and so we leave K-theory for now with the warning that here we have seen the definitions and nothing more.

4.2 Fredholm modules and K-homology

One of the central reasons that the techniques employed by Atiyah and Singer to compute the index of elliptic differential operators on manifolds continues to work for noncommutative spaces is the way K-theory enters the proof. Essentially K-theory, and the dual theory K-homology, make perfectly good sense for C^*-algebras, commutative or not. The best reference for K-homology is the book [HR], but [A, Ka1] are also worth a read in this context.

While we were very brief with K-theory, we will spend a little longer on K-homology as it is much closer to spectral triples. Indeed spectral triples are 'just' nice representatives of classes in K-homology.

For example, when we do cohomology on manifolds, we are very happy when we can represent an integral cohomology class by a differential form. This allows us to use calculus and geometry to make effective cohomological calculations. Spectral triples play an almost exactly analogous role.

Definition 4.3. *Let A be a separable C^*-algebra. A Fredholm module over A is given by a Hilbert space $\mathcal{H}$, a $*$-representation $\pi : A \to \mathcal{B}(\mathcal{H})$ and an operator $F : \mathcal{H} \to \mathcal{H}$ such that for all $a \in A$*

$$(F^2 - 1)\rho(a), \quad (F - F^*)\rho(a), \quad [F, \rho(a)] := F\rho(a) - \rho(a)F$$

are all compact operators. We say that $(\rho, \mathcal{H}, F)$ is even (or graded) if there is an operator $\gamma : \mathcal{H} \to \mathcal{H}$ such that $\gamma^2 = 1$, $\gamma = \gamma^*$, $\gamma F + F\gamma = 0$ and for all $a \in A$, $\gamma\rho(a) = \rho(a)\gamma$. Otherwise we call $(\rho, \mathcal{H}, F)$ odd.

We will usually consider algebras A which are unital and for which $\rho(1_A) = Id_{\mathcal{H}}$, and this simplifies the first two conditions on F: $F^2 - 1$ and $F - F^*$ are compact. In this case we have the following descriptions.

An odd Fredholm module is given by a (unital) representation ρ on $\mathcal{H}$ and an operator $F = 2P - 1 + K$ where K is compact and P is a projection commuting with $\rho(A)$ modulo compact operators

An even Fredholm module is given by a pair of representations ρ_+, ρ_- on Hilbert spaces $\mathcal{H}_+, \mathcal{H}_-$ respectively, and

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-, \quad \rho = \begin{pmatrix} \rho_+ & 0 \\ 0 & \rho_- \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & F_- \\ F_+ & 0 \end{pmatrix} \tag{4.1}$$

with $F_- = (F_+)^* + K$ with K compact. The conditions defining the Fredholm module tell us that $F_+ : \mathcal{H}_+ \to \mathcal{H}_-$ is a Fredholm operator.

Example 13. Let $\rho : \mathbb{C} \to \mathcal{B}(\mathcal{H})$ be the unique unital representation. Then an ungraded Fredholm module is given by an essentially self-adjoint Fredholm operator F. Likewise, a graded Fredholm module is given by an essentially self-adjoint Fredholm operator of the form (4.1).

For an even Fredholm module, we denote by $\mathrm{Index}(\rho, \mathcal{H}, F)$ the index of $F_+ : \mathcal{H}_1 \to \mathcal{H}_2$.

Example 14. Let $\mathcal{H} = L^2(S^1)$ and represent $C(S^1)$ on $\mathcal{H}$ as multiplication operators. So for $f \in C(S^1)$ and $\xi \in L^2(S^1)$, $(f\xi)(x) = f(x)\xi(x)$ for $x \in S^1$.

Let $P \in \mathcal{B}(\mathcal{H})$ be the projection onto $\overline{span}\{z^k : k \geq 0\}$. Since P is a projection, the operator $F = 2P - 1$ is self-adjoint and has square one. Thus to check that $(\mathcal{H}, F)$ (along with the multiplication representation) is an odd Fredholm module for $C(S^1)$, we just need to check that $[F, f]$ is compact for all $f \in C(S^1)$.

First, every $f \in C(S^1)$ is a norm (uniform) limit of trigonometric polynomials (Stone-Weierstrass) and so the norm limit of finite sums of z^k, $k \in \mathbb{Z}$, where $z : S^1 \to \mathbb{C}$ is the identity function.

Hence it suffices to show that $[F, z^k]$ is compact for each k, and so it is enough to show that $[P, z^k]$ is compact. Let $\xi \in \mathcal{H}$ so $\xi = \sum_{n\in\mathbb{Z}} c_n z^n$ (this sum converges in the Hilbert space norm). Then

$$Pz^k\xi = P\sum_{n\in\mathbb{Z}} c_n z^{n+k} = \sum_{n\geq -k} c_n z^{n+k}$$

while

$$z^k P\xi = \sum_{n\geq 0} c_n z^{n+k}.$$

The difference is

$$[P, z^k]\xi = \begin{cases} \sum_{n=-k}^{0} c_n z^{n+k} & k \geq 0 \\ \sum_{n=0}^{-k} c_n z^{n+k} & k \leq 0 \end{cases}$$

Hence $[P, z^k]$ is a rank k operator, and so compact.

The operators $T_f := PfP : P\mathcal{H} \to P\mathcal{H}$, $f \in C(S^1)$, are called Toeplitz or compression operators. One can show that

$$T_f^* = T_{\bar{f}} \quad \text{and} \quad T_f T_g = T_{fg} + K$$

where K is a compact operator. Composing with the quotient map $s : \mathcal{B}(P\mathcal{H}) \to \mathcal{Q}(P\mathcal{H})$ we get a $*$-homomorphism $C(S^1) \to \mathcal{Q}(P\mathcal{H})$ which is **faithful**!, see [HR] for instance. Hence we get an extension (short exact sequence)

$$0 \to \mathcal{K}(P\mathcal{H}) \to T \to C(S^1) \to 0$$

where T is the algebra generated by the T_f, $f \in C(S^1)$. This is called the Toeplitz extension.

Exercise What is the relationship between the Fredholm module for $C(S^1)$ in Example 14 and the spectral triple in Example 4? *Hint*: Look at the next example.

Example 15. Let $\mathcal{H} = L^2(\Lambda^* M, g)$ and let $C^\infty(M)$ act as multiplication operators. If $\mathcal{D} = d + d^*$, then we know that $(1 + \mathcal{D}^2)^{-1/2}$ is compact, and $F_\mathcal{D} = \mathcal{D}(1 + \mathcal{D}^2)^{-1/2}$ is bounded (by the functional calculus). Now we compute

$$F_\mathcal{D}^2 = \mathcal{D}^2(1 + \mathcal{D}^2)^{-1} = 1 - (1 + \mathcal{D}^2)^{-1}$$

which is the identity modulo compacts. Since $F_\mathcal{D}$ is self-adjoint and anticommutes with the grading γ of differential forms by degree, we need only check that $[F_\mathcal{D}, f]$ is compact for all $f \in C^\infty(M)$. So

$$[F_\mathcal{D}, f] = [\mathcal{D}, f](1 + \mathcal{D}^2)^{-1/2} - \mathcal{D}(1 + \mathcal{D}^2)^{-1/2}[(1 + \mathcal{D}^2)^{1/2}, f](1 + \mathcal{D}^2)^{-1/2}.$$

Since $[\mathcal{D}, f]$ is Clifford multiplication by df, the first term is compact. Likewise, the second term will be compact if we can see that $[(1 + \mathcal{D}^2)^{1/2}, f]$ is bounded. But the symbol of f is fId, so the order of the commutator is $1 + 0 - 1 = 0$, and so we have a bounded pseudodifferential operator. Hence we get an even Fredholm module for the algebra $C^\infty(M)$.

Definition 4.4. *Let $(\rho, \mathcal{H}, F)$ be a Fredholm module, and suppose that $U : \mathcal{H}' \to \mathcal{H}$ is a unitary. Then $(U^*\rho U, \mathcal{H}', U^*FU)$ is also a Fredholm module (with grading $U^*\gamma U$ if γ is a grading of $(\rho, \mathcal{H}, F)$) and we say that it is unitarily equivalent to $(\rho, \mathcal{H}, F)$.*

Definition 4.5. *Let $(\rho, \mathcal{H}, F_t)$ be a family of Fredholm modules parameterised by $t \in [0,1]$ with $\rho, \mathcal{H}$ constant. If the function $t \to F_t$ is norm continuous, we call this family an operator homotopy between $(\rho, \mathcal{H}, F_0)$ and $(\rho, \mathcal{H}, F_1)$, and say that these two Fredholm modules are operator homotopic.*

If $(\rho_1, \mathcal{H}_1, F_1)$ and $(\rho_2, \mathcal{H}_2, F_2)$ are Fredholm modules over the same algebra A, then $(\rho_1 \oplus \rho_2, \mathcal{H}_1 \oplus \mathcal{H}_2, F_1 \oplus F_2)$ is a Fredholm module over A, called the direct sum.

Definition 4.6. *Let $p = 0, 1$. The K-homology group $K^p(A)$ is the abelian group with one generator $[x]$ for each unitary equivalence class of Fredholm modules (even or graded if $p = 0$, and odd or ungraded for $p = 1$) with the following relations:*

1) If x_0 and x_1 are operator homotopic Fredholm modules (both even or both odd) then $[x_0] = [x_1]$ in $K^p(A)$, and

2) If x_0 and x_1 are two Fredholm modules (both even or both odd) then $[x_0 \oplus x_1] = [x_0] + [x_1]$ in $K^p(A)$.

The zero element is the class of the zero module, which is the zero Hilbert space, zero representation and naturally a zero operator. There are other representatives of this class also which we require in order to be able to display inverses.

Definition 4.7. *A Fredholm module $(\rho, \mathcal{H}, F)$ is called degenerate if $F = F^*$, $F^2 = 1$ and $[F, \rho(a)] = 0$ for all $a \in A$.*

Exercise The class of a degenerate module is zero in K-homology. *Hint*: Consider $\oplus^\infty(\rho, \mathcal{H}, F)$ and $(\rho, \mathcal{H}, F) \oplus \oplus^\infty(\rho, \mathcal{H}, F)$.

Lemma 4.8. *[HR] If $x = (\rho, \mathcal{H}, F)$ is an odd Fredholm module, then the class of $-[x]$ is represented by the Fredholm module $(\rho, \mathcal{H}, -F)$. For an even Fredholm module $x = (\rho, \mathcal{H}, F, \gamma)$ the inverse is represented by $(\rho, \mathcal{H}, -F, -\gamma)$.*

Proof. We do the even case, by showing that

$$\left(\begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix}, \mathcal{H} \oplus \mathcal{H}, \begin{pmatrix} F & 0 \\ 0 & -F \end{pmatrix}, \begin{pmatrix} \gamma & 0 \\ 0 & -\gamma \end{pmatrix} \right)$$

is operator homotopic to the degenerate module

$$\left(\begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix}, \mathcal{H} \oplus \mathcal{H}, \begin{pmatrix} 0 & Id_{\mathcal{H}} \\ Id_{\mathcal{H}} & 0 \end{pmatrix}, \begin{pmatrix} \gamma & 0 \\ 0 & -\gamma \end{pmatrix} \right).$$

We do this by displaying the homotopy

$$F_t = \begin{pmatrix} \cos(\pi t/2) F & \sin(\pi t/2) Id_{\mathcal{H}} \\ \sin(\pi t/2) Id_{\mathcal{H}} & -\cos(\pi t/2) F \end{pmatrix}.$$

We leave the details as an **Exercise**. □

Let $\psi : A \to B$ be a $*$-homomorphism, and $(\rho, \mathcal{H}, F)$ a Fredholm module over B. Then $(\rho \circ \psi, \mathcal{H}, F)$ is a Fredholm module over A. This allows us to define

$$\psi^* : K^*(B) \to K^*(A) \quad \text{by} \quad \psi^*[(\rho, \mathcal{H}, F)] = [(\rho \circ \psi, \mathcal{H}, F)]$$

and so K-homology is a contravariant functor from (separable) C^*-algebras to abelian groups. We denote by $K^*(A) = K^0(A) \oplus K^1(A)$.

Being able to work modulo compact operators gives us plenty of freedom. Sometimes however, it is better to have 'nice' representatives of K-homology classes.

Lemma 4.9. *[HR] Every K-homology class in $K^*(A)$ can be represented by a Fredholm module $(\rho, \mathcal{H}, F)$ with $F = F^*$ and $F^2 = 1$.* **Alternatively***, we may suppose that $(\rho, \mathcal{H}, F)$ is nondegenerate in the sense that $\rho(A)\mathcal{H}$ is dense in $\mathcal{H}$. In general we can not do both these things at the same time.*

We will call any Fredholm module with $F = F^*$ and $F^2 = 1$ a **normalised Fredholm module**. In [HR], this is called an involutive Fredholm module.

Usually, we will omit the representation, and refer to a Fredholm module $(\mathcal{H}, F)$ for a C^*-algebra A.

4.3 The index pairing

The pairing between K-theory and K-homology is given in terms of the Fredholm index. First, we recall a useful trick. If $(\mathcal{H}, F)$ is a Fredholm module for an algebra $\mathcal{A}$, then $(\mathcal{H}^k, F \otimes Id_k)$ is a Fredholm module for $M_k(\mathcal{A})$. If $(\mathcal{H}, F)$ is normalised so is $(\mathcal{H}^k, F \otimes Id_k)$. We leave this as an **Exercise**.

Let $(\mathcal{H}, F, \gamma)$ be an even Fredholm module for an algebra $\mathcal{A}$ and $p \in M_k(\mathcal{A})$ a projection. Then the pairing between $[p] \in K_0(\mathcal{A})$ and $[(\mathcal{H}, F, \gamma)] \in K^0(\mathcal{A})$ is given by

$$\text{Index}(p(F^+ \otimes Id_k)p : p\mathcal{H}^k \to p\mathcal{H}^k).$$

When $(\mathcal{H}, F)$ is an odd Fredholm module over $\mathcal{A}$, and $u \in M_k(\mathcal{A})$ is a unitary, the pairing between $[u] \in K_1(\mathcal{A})$ and $[(\mathcal{H}, F)] \in K^1(\mathcal{A})$ is given by

$$\text{Index}(P_k u P_k - (1 - P_k) : \mathcal{H}^k \to \mathcal{H}^k)$$

where $P_k = \frac{1}{2}(1 + F) \otimes Id_k$.

Exercise Show that these two indices are well defined. That is, show that pF^+p and PuP are Fredholm.

When P is a projection (say when $F = F^*$ and $F^2 = 1$), operators like PuP are called Toeplitz operators, and sometimes one speaks about the compression of u to $P\mathcal{H}$. We won't discuss it at length, but the index of PuP is also equal to the net amount of spectrum crossing zero from negative to positive along the path $(1-t)P + tPuP$ as t goes from 0 to 1. This **spectral flow** has a rigorous analytic definition due to Phillips, [Ph1, Ph2], which we won't pursue here. Other more topological definitions go back to Atiyah and Singer.

Example 16. Let $\mathcal{D}^+ : \Gamma(E) \to \Gamma(F)$ be a first order elliptic differential operator on the manifold M. Let $(C^\infty(M), L^2(E) \oplus L^2(F), \mathcal{D})$ be the even spectral triple with $\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}^- \\ \mathcal{D}^+ & 0 \end{pmatrix}$ where $\mathcal{D}^- = (\mathcal{D}^+)^*$.

If W is another vector bundle, we can associate to it a projection $p \in M_N(C^\infty(M))$ so that

$$L^2(E \otimes W) = pL^2(E)^N,$$

and similarly for $F \otimes W$. Then $\mathcal{D} \otimes_\nabla Id_W = p(\mathcal{D} \otimes 1_N)p$ and we see that

$$\text{Index}((\mathcal{D} \otimes_\nabla Id_W)^+) = \text{Index}(p(\mathcal{D}^+ \otimes 1_N)p).$$

Exercise Prove the equalities in Example 16. *Hint*: If $\Gamma(E) = pC^\infty(M)^N$ then the composition

$$pC^\infty(M)^N \xrightarrow{i} C^\infty(M)^N \xrightarrow{d} \Gamma(T^*M)^N \xrightarrow{p} \Gamma(E \otimes T^*M)$$

is a connection.

From now on we will assume that any projection or unitary we want to pair a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ with lives in the algebra $\mathcal{A}$ rather than $M_k(\mathcal{A})$.

4.4 The index pairing for finitely summable Fredholm modules

In this section we briefly describe how we can compute the index pairing for (the class of) certain special Fredholm modules. Whilst special, these are the kinds of Fredholm modules one often encounters in nature.

Before stating the definition, we need a couple of extras. The first is that we must abandon C^*-algebras. The problem with C^*-algebras can be seen in the commutative case: if we want to start computing index $p\mathcal{D}p$ where $\mathcal{D}$ is a first order elliptic differential operator, we are going to require that p be at least C^1. That means we can not allow ourselves to use any old representative of a K-theory class, but something a bit smoother. We will return to this point.

The other thing we require is the definition of the Schatten ideals.

Definition 4.10. *For any $p \geq 1$, define the p-th Schatten ideal of the Hilbert space $\mathcal{H}$ to be*

$$\mathcal{L}^p(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : \text{Trace}(|T|^p) < \infty\}, \quad |T| = \sqrt{T^*T}. \tag{4.2}$$

Remarks 1) These ideals are all two-sided, but are not norm closed. As the compact operators are the only norm closed ideal of the bounded operators on Hilbert space, all the $\mathcal{L}^p$ have norm closure equal to the compact operators.

2) The ideal $\mathcal{L}^2(\mathcal{H})$ is called the Hilbert-Schmidt class, and is a Hilbert space for the inner product

$$(T, S) = \text{Trace}(S^*T) \quad \forall T, S \in \mathcal{L}^2(\mathcal{H}). \tag{4.3}$$

In particular, it is complete for the norm $\| T \|_2 = \text{Trace}(T^*T)^{1/2}$.

3) More generally, $\mathcal{L}^p(\mathcal{H})$ is complete for the norm

$$\| T \|_p := \mathrm{Trace}(|T|^p)^{1/p}, \; T \in \mathcal{L}^p(\mathcal{H}). \tag{4.4}$$

Moreover, if $B, C \in \mathcal{B}(\mathcal{H})$ and $T \in \mathcal{L}^p(\mathcal{H})$, then

$$\| BTC \|_p \leq \| B \| \| T \|_p \| C \|, \tag{4.5}$$

where $\| \cdot \|$ is the usual operator norm. The fancy language for such an ideal is a 'symmetrically normed ideal', for obvious reasons.

4) Two useful (and immediate) facts are

a) If $T \in \mathcal{L}^p(\mathcal{H})$ then $T^p \in \mathcal{L}^1(\mathcal{H})$.

b) If $T_i \in \mathcal{L}^{p_i}(\mathcal{H})$ then $T_1 \cdots T_k \in \mathcal{L}^p(\mathcal{H})$ where

$$\frac{1}{p} = \sum_{j=1}^{k} \frac{1}{p_j}. \tag{4.6}$$

This last can be shown using the Hölder inequality. Note the analogy with the L^p spaces of classical analysis.

After this very brief summary of these operator ideals, we make the following definition.

Definition 4.11 (Connes)**.** *Let $\mathcal{A}$ be a unital $*$-algebra. A (normalised) Fredholm module $(\mathcal{H}, F)$ for $\mathcal{A}$ is $p+1$-summable, $p \in \mathbb{N}$, if for all $a \in \mathcal{A}$ we have*

$$[F, a] \in \mathcal{L}^{p+1}(\mathcal{H}).$$

Example 17. When we construct the Hodge-de Rham Fredholm module, instead of starting with $d + d^*$, we can start with

$$\mathcal{D}_m = \begin{pmatrix} d + d^* & m \\ m & -(d + d^*) \end{pmatrix}, \quad m > 0,$$

acting on $\mathcal{H}_2 = L^2(\Lambda^* M, g) \oplus L^2(\Lambda^* M, g)$ with the grading $\gamma \oplus -\gamma$. The representation of $C^\infty(M)$ on $\mathcal{H}_2$ is as multiplication operators in the first copy, and by zero in the second. Since $\mathcal{D}_m$ is invertible, we are free to define $F_{\mathcal{D}_m} = \mathcal{D}_m |\mathcal{D}_m|^{-1}$. Again we wind up with a Fredholm module, but this time $F^2_{\mathcal{D}_m} = Id_{\mathcal{H}_2}$, and so we have a normalised Fredholm module.

From Weyl's Theorem, $|\mathcal{D}_m|^{-1}$ has eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots$ satisfying

$$\lambda_n = C\, n^{-1/\dim M} + o(n^{-1/\dim M}).$$

From this and the standard commutator tricks, it is easy to see that $(\mathcal{H}_2, F_{\mathcal{D}_m})$ is $\dim(M) + 1$ summable.

For finitely summable normalised Fredholm modules we can define cyclic cocycles whose class in periodic cyclic cohomology is called the **Chern character**. There is some more information about this in Appendix B, but here is the definition and result which make it worth studying.

Definition 4.12 (Connes). *Let $(\mathcal{H}, F)$ be a $p+1$-summable normalised Fredholm module for the $*$-algebra $\mathcal{A}$. For any $n \geq p$ of the same parity as the Fredholm module we define cyclic cocycles by*

$$Ch_n(\mathcal{H}, F)(a_0, a_1 \cdots, a_n) = \frac{\lambda_n}{2} \operatorname{Trace}(\gamma[F, a_0][F, a_1] \cdots [F, a_n]),$$

where γ is 1 if the module is odd, and the normalisation constants are

$$\lambda_n = \begin{cases} (-1)^{n(n-1)/2}\Gamma(n/2+1) & (even) \\ \sqrt{2i}(-1)^{n(n-1)/2}\Gamma(n/2+1) & (odd) \end{cases}$$

The Chern character $Ch_*(\mathcal{H}, F)$ is the class of these cocycles in periodic cyclic cohomology.

For $T \in \mathcal{B}(\mathcal{H})$ such that $FT + TF \in \mathcal{L}^1(\mathcal{H})$, define the 'conditional trace'

$$\operatorname{Trace}'(T) = \frac{1}{2} \operatorname{Trace}(F(FT + TF)).$$

Note that if $T \in \mathcal{L}^1(\mathcal{H})$ then $\operatorname{Trace}'(T) = \operatorname{Trace}(T)$. Then define

$$\operatorname{Trace}_s(T) = \operatorname{Trace}'(\gamma T).$$

Here $\gamma = Id_{\mathcal{H}}$ if n is odd. Then we can write

$$Ch_n(\mathcal{H}, F, \gamma)(a_0, a_1, \ldots, a_n) = \lambda_n \operatorname{Trace}_s(a_0[F, a_1] \cdots [F, a_n]). \quad (4.7)$$

Theorem 4.13 (Connes). *Let $(\mathcal{H}, F)$ be a finitely summable normalised Fredholm module over $\mathcal{A}$. Then for any $[e] \in K_0(\mathcal{A})$*

$$\langle [e], [(\mathcal{H}, F)] \rangle = Ch_*(\mathcal{H}, F)(e) := \frac{1}{(n/2)!} Ch_n(\mathcal{H}, F)(e, e, \ldots, e)$$

for n large enough and even. For $[u] \in K_1(\mathcal{A})$

$$\langle [u], [(\mathcal{H}, F)] \rangle = Ch_*(\mathcal{H}, F)(u) := \frac{1}{\sqrt{2i}2^n\Gamma(n/2+1)} Ch_n(\mathcal{H}, F)(u^*, u, \ldots, u)$$

for n large enough and odd.

Proof. We will prove this in the even case using 'bare hands'. So we can take $(\mathcal{H}, F, \gamma)$ to be given by

$$F = \begin{pmatrix} 0 & Q \\ P & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$$

where $\mathcal{H}^\pm = \frac{1 \pm \gamma}{2}\mathcal{H}$, as usual, and we also have $PQ = Id_{\mathcal{H}^-}$ and $QP = Id_{\mathcal{H}^+}$. We suppose that $(\mathcal{H}, F, \gamma)$ is $p+1$ summable, and we use the representative of the Chern character with $n = 2k$, $n \geq p+1$.

Now we can find a representative $e \in [e]$ with $e \in M_m(\mathcal{A})$ for some $m \geq 1$. Using Morita invariance, we see that with no loss of generality we may take $e \in \mathcal{A}$. With these conventions, we wish to compute

$$\operatorname{Index}(ePe) : e\mathcal{H}^+ \to e\mathcal{H}^-.$$

Set $Q' = eQe$ and $P' = ePe$. Then Q' is a parametrix for P' and

$$1 - P'Q' : e\mathcal{H}^- \to e\mathcal{H}^-$$

$$1 - Q'P' : e\mathcal{H}^+ \to e\mathcal{H}^+$$

are both compact, and

$$e - eFeFe = \begin{pmatrix} 1 - Q'P' & 0 \\ 0 & 1 - P'Q' \end{pmatrix} = \begin{pmatrix} e - eQePe & 0 \\ 0 & e - ePeQe \end{pmatrix} \text{ on } e\mathcal{H}^+ \oplus e\mathcal{H}^-.$$

As $Q^* = P \Rightarrow Q'^* = P'$, we see that

$$\ker Q' = \operatorname{coker} P' \quad \operatorname{coker} Q' = \ker P'$$

which gives

$$Q'P' : (\ker P')^\perp \to (\ker P')^\perp$$
$$P'Q' : (\operatorname{coker} P')^\perp \to (\operatorname{coker} P')^\perp.$$

This is enough to show that

$$\operatorname{Index}(P') = \operatorname{Trace}(1 - Q'P') - \operatorname{Trace}(1 - P'Q')$$

but we wish to be explicit. Let $\psi \in (\ker P')^\perp \subset e\mathcal{H}^+$ and compute

$$\begin{aligned} eQePe\psi &= eQeP\psi \\ &= -[Q,e]eP\psi + QeP\psi \\ &= [Q,e][P,e]\psi + [Q,e]P\psi - [Q,e]Pe\psi + e\psi \\ &= (1 + [Q,e][P,e])\psi. \end{aligned} \tag{4.8}$$

For $\psi \in \ker P'$,

$$(1 - Q'P')\psi = \psi \tag{4.9}$$

and for $\psi \in \operatorname{coker} P'$,

$$(1 - P'Q')\psi = \psi. \tag{4.10}$$

Thus

$$\begin{pmatrix} e - eQePe & 0 \\ 0 & e - ePeQe \end{pmatrix} = \begin{pmatrix} [Q,e][P,e] & 0 & 0 & 0 \\ 0 & 1_{\ker P'} & 0 & 0 \\ 0 & 0 & [P,e][Q,e] & 0 \\ 0 & 0 & 0 & 1_{\operatorname{coker} P'} \end{pmatrix}. \tag{4.11}$$

Also, for $k \geq \frac{p+1}{2}$, the operators

$$([Q,e][P,e])^k \text{ and } ([P,e][Q,e])^k \tag{4.12}$$

are trace class. Moreover

$$\begin{aligned} \operatorname{Trace}(([P,e][Q,e])^k) &= \operatorname{Trace}([P,e][Q,e] \cdots [P,e][Q,e]) \\ &= \operatorname{Trace}([Q,e][P,e] \cdots [Q,e][P,e]) \\ &= \operatorname{Trace}(([Q,e][P,e])^k) \end{aligned}$$

where we have used the cyclicity of the trace. So

$$\begin{aligned}
\mathrm{Trace}(\gamma(e-eFeFe)^k) &= \mathrm{Trace}\left(\gamma\begin{pmatrix} e-eQePe & 0 \\ 0 & e-ePeQe\end{pmatrix}^k\right)\\
&= \mathrm{Trace}\left(\begin{pmatrix} [Q,e][P,e] & 0 & 0 & 0\\ 0 & 1_{\ker P'} & 0 & 0\\ 0 & 0 & -[P,e][Q,e] & 0\\ 0 & 0 & 0 & -1_{\mathrm{coker}\, P'}\end{pmatrix}^k\right)\\
&= \dim\ker P' - \dim\mathrm{coker}\, P'\\
&= \mathrm{Index}\, P'\\
&= \mathrm{Index}\, ePe. \qquad (4.13)
\end{aligned}$$

So for $k \geq \frac{p+1}{2}$ we have

$$\mathrm{Index}(ePe) = \mathrm{Trace}(\gamma(e-eFeFe)^k).$$

However,

$$\begin{aligned}
e - eFeFe &= e - e[F,e]Fe - e\\
&= -e[F,e][F,e]e - e[F,e]eF.
\end{aligned}$$

Using $e = e^2$, we see that (for any derivation d) $de = de^2 = e(de) + (de)e$. Multiplying on the right and left by e gives $e(de)e = 2e(de)e$, whence $e(de)e = 0$. So

$$e - eFeFe = -e[F,e]^2 e.$$

Thus

$$\begin{aligned}
\mathrm{Index}(ePe) &= (-1)^k\,\mathrm{Trace}(\gamma(e[F,e]^2e)^k)\\
&= (-1)^k\,\mathrm{Trace}(\gamma e[F,e]^{2k})\\
&= \frac{1}{k!}(-1)^k Ch_*(\mathcal{H},F,\gamma)(e,e,\ldots,e),
\end{aligned}$$

where in the second last line we used the facts $e[F,e]^2 = [F,e]^2e$, $e\gamma = \gamma e$ and the cyclicity of the trace, while in the last line we used the fact that $\mathrm{Trace}'(T) = \mathrm{Trace}(T)$ for $T \in \mathcal{L}^1(\mathcal{H})$. □

Exercise Do the odd case using 'bare hands'.

The pairing only depends on the K-theory class of e and the K-homology class of $(\mathcal{H}, F)$. Whilst the Chern character in this form is very useful for proving basic facts about the index pairing, and relating it to cyclic cohomology, it is not the most computable form for examples.

Imagine trying to compute the pairing of the Hodge-de Rham Fredholm module with K-theory this way. First take your Hilbert space realisation of $d + d^*$, form an invertible version

$$\mathcal{D}_m = \begin{pmatrix} d+d^* & m\\ m & -(d+d^*)\end{pmatrix},$$

and then take the phase $F = \mathcal{D}_m|\mathcal{D}_m|^{-1}$. This would appear to be much harder to compute, and compute with, than simply differentiating functions. **The issue of computability is the most fundamental reason for being interested in spectral triples.**

Chapter 5

Spectral triples and computing the index pairing

5.1 Smoothness of spectral triples and algebras

If we want to be able to express the pairing of the K-homology class $[(\mathcal{A},\mathcal{H},\mathcal{D})]$ with K-theory directly in terms of $(\mathcal{A},\mathcal{H},\mathcal{D})$, we need more assumptions. In particular, to compare our computations with the Chern character computations, we will need to know that $(\mathcal{H},\mathcal{D}(1+\mathcal{D}^2)^{-1/2})$ is a finitely summable Fredholm module.

Smoothness (aka regularity) is about having sufficient 'quantum differentiability' for elements of our algebra. However we only have a formula for the pairing for finitely summable Fredholm modules. To ensure that a spectral triple represents a K-homology class with a finitely summable representative, we need a summability assumption on the spectral triple, and some smoothness as well. The interplay between smoothness(=differentiability) and summability(=dimension) is more complicated than in the commutative case.

Definition 5.1. *A spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D})$ is QC^k for $k\geq 1$ (Q for quantum) if for all $a\in\mathcal{A}$ the operators a and $[\mathcal{D},a]$ are in the domain of δ^k, where $\delta(T)=[|\mathcal{D}|,T]$ is the partial derivation on $\mathcal{B}(\mathcal{H})$ defined by $|\mathcal{D}|$. We say that $(\mathcal{A},\mathcal{H},\mathcal{D})$ is QC^∞ if it is QC^k for all $k\geq 1$.*

Remark. The notation is meant to be analogous to the classical case, but we introduce the Q so that there is no confusion between quantum differentiability of $a\in\mathcal{A}$ and classical differentiability of functions.

Remarks concerning derivations and commutators. By partial derivation we mean that δ is defined on some subalgebra of $\mathcal{B}(\mathcal{H})$ which need not be (weakly) dense in $\mathcal{B}(\mathcal{H})$. More precisely, $\mathrm{dom}\delta=\{T\in\mathcal{B}(\mathcal{H}):\delta(T)\text{ is bounded}\}$. We also note that if $T\in\mathcal{B}(\mathcal{H})$, one can show that $[|\mathcal{D}|,T]$ is bounded if and only if $[(1+\mathcal{D}^2)^{1/2},T]$ is bounded, by using the functional calculus to show that $|\mathcal{D}|-(1+\mathcal{D}^2)^{1/2}$ extends to a bounded operator in $\mathcal{B}(\mathcal{H})$. In fact, writing $|\mathcal{D}|_1=(1+\mathcal{D}^2)^{1/2}$ and $\delta_1(T)=[|\mathcal{D}|_1,T]$ we have

$$\mathrm{dom}\delta^n=\mathrm{dom}\delta_1^n\quad\forall n.$$

Thus the condition defining QC^∞ can be replaced by

$$a, [\mathcal{D}, a] \in \bigcap_{n\geq 0} \mathrm{dom}\delta_1^n \quad \forall a \in \mathcal{A}.$$

This is important in situations where we cannot assume $|\mathcal{D}|$ is invertible.

Proposition 5.2. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a QC^1 spectral triple, and define $F_\mathcal{D} = \mathcal{D}(1+\mathcal{D}^2)^{-1/2}$. Then $(\mathcal{H}, F_\mathcal{D})$ is a Fredholm module for the C^*-algebra $A := \overline{\mathcal{A}}$.*

Proof. For $a \in \mathcal{A}$ we have

$$\begin{aligned} [F_\mathcal{D}, a] &= [\mathcal{D}, a](1+\mathcal{D}^2)^{-1/2} + \mathcal{D}[(1+\mathcal{D}^2)^{-1/2}, a] \\ &= [\mathcal{D}, a](1+\mathcal{D}^2)^{-1/2} - \mathcal{D}(1+\mathcal{D}^2)^{-1/2}[(1+\mathcal{D}^2)^{1/2}, a](1+\mathcal{D}^2)^{-1/2}. \end{aligned}$$

This is a compact operator. If $\{a_k\}_{k\geq 0} \subset \mathcal{A}$ is a sequence converging in (operator) norm then

$$\|[F_\mathcal{D}, a_k - a_m]\| \leq 2\|F_\mathcal{D}\| \, \|a_k - a_m\| \leq 2\|a_k - a_m\| \to 0.$$

Hence if $a = \lim_k a_k$ with $a_k \in \mathcal{A}$ and convergence in norm,

$$[F_\mathcal{D}, a] = \lim_k [F_\mathcal{D}, a_k]$$

and this is a limit of compact operators, and so compact. □

Thus every QC^1 spectral triple defines a K-homology class. In order that this spectral triple defines a finitely summable Fredholm module, and so a Chern character, we need finite summability of the spectral triple.

In fact a QC^0 spectral triple defines a Fredholm module, but the proof is more involved. See [CP1] for a proof.

We finish this section with a couple of definitions and results about the kinds of algebras which arise in the company of spectral triples.

Definition 5.3. *A Fréchet algebra is a locally convex, metrizable and complete topological vector space with jointly continuous multiplication.*

We will always suppose that we can define the Fréchet topology of $\mathcal{A}$ using a countable collection of submultiplicative seminorms which includes the C^*-norm of $\overline{\mathcal{A}} = A$, and note that the multiplication is jointly continuous. By replacing any seminorm q by $\frac{1}{2}(q(a) + q(a^*))$, we may suppose that $q(a) = q(a^*)$ for all $a \in \mathcal{A}$.

Definition 5.4. *A subalgebra $\mathcal{A}$ of a C^*-algebra A is a* **pre-C^*-algebra** *or* **stable under the holomorphic functional calculus** *if whenever $a \in \mathcal{A}$ is invertible in A, it is invertible in $\mathcal{A}$. Equivalently, $\mathcal{A}$ is a pre-C^*-algebra if whenever $f : \mathbb{C} \to \mathbb{C}$ is a function holomorphic in a neighbourhood of the spectrum of $a \in \mathcal{A}$, then the element $f(a) \in A$ defined by the continuous functional calculus is in fact in $\mathcal{A}$, i.e. $f(a) \in \mathcal{A}$.*

Definition 5.5. *A $*$-algebra $\mathcal{A}$ is smooth if it is Fréchet and $*$-isomorphic to a proper dense subalgebra $i(\mathcal{A})$ of a C^*-algebra A which is stable under the holomorphic functional calculus.*

Thus saying that $\mathcal{A}$ is smooth means that $\mathcal{A}$ is Fréchet and a pre-C^*-algebra. Asking for $i(\mathcal{A})$ to be a *proper* dense subalgebra of A immediately implies that the Fréchet topology of $\mathcal{A}$ is finer than the C^*-topology of A (since Fréchet means locally convex, metrizable and complete.)

It has been shown that if $\mathcal{A}$ is smooth in A then $M_n(\mathcal{A})$ is smooth in $M_n(A)$, [GVF, Sc]. This ensures that the K-theories of the two algebras are isomorphic, the isomorphism being induced by the inclusion map i. This definition ensures that a smooth algebra is a 'good' algebra, [GVF], so these algebras have a sensible spectral theory which agrees with that defined using the C^*-closure, and the group of invertibles is open.

Lemma 5.6. *[[GVF, R2]] If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a QC^∞ spectral triple, then $(\mathcal{A}_\delta, \mathcal{H}, \mathcal{D})$ is also a QC^∞ spectral triple, where $\mathcal{A}_\delta$ is the completion of $\mathcal{A}$ in the locally convex topology determined by the seminorms*

$$q_{ni}(a) = \| \delta^n d^i(a) \|, \quad n \geq 0, \ i = 0, 1,$$

where $d(a) = [\mathcal{D}, a]$. Moreover, $\mathcal{A}_\delta$ is a smooth algebra.

Thus whenever we have a QC^∞ spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, we may suppose without loss of generality that the algebra $\mathcal{A}$ is a Fréchet pre-C^*-algebra. Thus $\mathcal{A}$ suffices to capture all the K-theory of A. This is necessary if we are to use spectral triples to compute the index pairing.

A QC^∞ spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ for which $\mathcal{A}$ is complete has not only a holomorphic functional calculus for $\mathcal{A}$, but also a C^∞ *functional calculus* for selfadjoint elements: we quote [R2, Prop. 22].

Proposition 5.7 (C^∞ Functional Calculus)**.** *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a QC^∞ spectral triple, and suppose $\mathcal{A}$ is complete. Let $f : \mathbb{R} \to \mathbb{C}$ be a C^∞ function in a neighbourhood of the spectrum of $a = a^* \in \mathcal{A}$. If we define $f(a) \in A$ using the continuous functional calculus, then in fact $f(a)$ lies in $\mathcal{A}$.* □

Remark For each $a = a^* \in \mathcal{A}$, the C^∞-functional calculus defines a continuous homomorphism $\Psi : C^\infty(U) \to \mathcal{A}$, where $U \subset \mathbb{R}$ is any open set containing the spectrum of a, and the topology on $C^\infty(U)$ is that of uniform convergence of all derivatives on compact subsets.

The following proposition extends this result to the case of smooth functions of several variables, yielding a *multivariate C^∞ functional calculus* (proved in [RV]). Before stating it, we recall the continuous functional calculus for a finite set $a_1, \ldots, a_n$ of commuting selfadjoint elements of a unital C^*-algebra A. These generate a unital $*$-algebra whose closure in A is a C^*-subalgebra $C^*(1, a_1, \ldots, a_n)$; let Δ be its (compact) space of characters. Evaluation of polynomials $p \mapsto p(a_1, \ldots, a_n)$ yields a surjective morphism from $C\big(\prod_{j=1}^n \mathrm{spec}(a_j)\big)$ onto $C^*(1, a_1, \ldots, a_n) \simeq C(\Delta)$ which corresponds, via the Gelfand functor, to a continuous injection $\Delta \hookrightarrow \prod_{j=1}^n \mathrm{spec}(a_j)$; this joint spectrum Δ may thus be regarded as a compact subset of $\mathbb{R}^n$. If $h \in C(\Delta)$, we may define $h(a_1, \ldots, a_n)$ as the image of $h|_\Delta$ in $C^*(1, a_1, \ldots, a_n)$ under the Gelfand isomorphism.

Proposition 5.8. *([RV]) Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a QC^∞ spectral triple. Let $a_1, \ldots, a_n$ be mutually commuting selfadjoint elements of $\mathcal{A}$, and let $\Delta \subset \mathbb{R}^n$ be their joint spectrum. Let $f : \mathbb{R}^n \to \mathbb{C}$ be a C^∞ function supported in a bounded open neighbourhood U of Δ. Then $f(a_1, \ldots, a_n)$ lies in $\mathcal{A}_\delta$.*

Proof. We first define the operator $f(a_1, \ldots, a_n)$ lying in A, the C^*-completion of $\mathcal{A}$, using the continuous functional calculus.

Since f is a compactly supported smooth function on $\mathbb{R}^n$, we may alternatively define $f(a_1, \dots, a_n) \in A$ by a Fourier integral:

$$f(a_1, \dots, a_n) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \hat{f}(s_1, \dots, s_n) \exp(i\, s \cdot a)\, d^n s, \tag{5.1}$$

where $s \cdot a = s_1 a_1 + \cdots + s_n a_n$. Since δ (and likewise $d := [\mathcal{D}, \cdot])$ is a norm-closed derivation from $\mathcal{A}$ to $\mathcal{B}(\mathcal{H})$, we may conclude that $f(a_1, \dots, a_n) \in \mathrm{Dom}\delta$ with

$$\delta(f(a_1, \dots, a_n)) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \hat{f}(s_1, \dots, s_n) \delta(\exp(i\, s \cdot a))\, d^n s, \tag{5.2}$$

provided we can establish dominated convergence for the integral on the right hand side [BR]. Just as in the one-variable case [R2], since each $a_j \in \mathrm{Dom}\delta$, we find that $\exp(i\, s \cdot a) = \prod_j \exp(is_j a_j)$ lies in $\mathrm{Dom}\delta$ also:

$$\delta(\exp(is_j a_j)) = is_j \int_0^1 \exp(its_j a_j)\, \delta(a_j)\, \exp(i(1-t)s_j a_j)\, dt, \tag{5.3}$$

and in particular,

$$\|\delta(\exp(i\, s \cdot a))\| \le C \sum_j |s_j|, \qquad C = \max_j \Bigl(\|\delta(a_j)\| \prod_{i \neq j} \|a_i\| \Bigr).$$

A norm bound which dominates the right hand side of (5.2) is thus given by

$$\int_{\mathbb{R}^n} |\hat{f}(s_1, \dots, s_n)|\, \|\delta(\exp(i\, s \cdot a))\|\, d^n s \le C \sum_{j=1}^n (2\pi)^{-n/2} \int_{\mathbb{R}^n} |\hat{f}(s_1, \dots, s_n)|\, |s_j|\, d^n s.$$

Let A_1 be the completion of $\mathcal{A}$ for the norm $\|a\|_{\mathcal{D}} := \|a\| + \|da\|$; notice that $A_1 \subseteq A$. Replacing δ by d in the previous argument, we find that

$$\|f(a_1, \dots, a_n)\|_{\mathcal{D}} \le \|\hat{f}\|_1 + \|da\| \sum_{j=1}^n (2\pi)^{-n/2} \int_{\mathbb{R}^n} |\hat{f}(s_1, \dots, s_n)|\, |s_j|\, d^n s.$$

Therefore, $f(a_1, \dots, a_n)$ can be approximated, in the $\|\cdot\|_{\mathcal{D}}$ norm, by Riemann sums for (5.1) belonging to $\mathcal{A}$, and thus $f(a_1, \dots, a_n) \in A_1$.

Since δ and d are commuting derivations, we obtain that $\delta(f(a_1, \dots, a_n)) \in \mathrm{Dom}d$ and $d(f(a_1, \dots, a_n)) \in \mathrm{Dom}\delta$ for $a \in \mathcal{A}$, and $\|\delta(d(f(a_1, \dots, a_n)))\|$ is bounded by a linear combination of expressions $\|\delta(da_j)\| \int |\hat{f}(s_1, \dots, s_n)|\, |s_j|\, d^n s$ and $\|\delta a_j\|\, \|da_k\| \int |\hat{f}(s_1, \dots, s_n)|\, |s_j s_k|\, d^n s$. In particular, $\|\delta(f(a_1, \dots, a_n))\|_{\mathcal{D}}$ also has a bound of this type.

For each $m = 1, 2, 3, \dots$, let A_m be the completion of $\mathcal{A}$ for the norm $\sum_{k \le m} \|\delta^k(a)\|_{\mathcal{D}}$. Then δ extends to a norm-closed derivation from A_m to $\mathcal{B}(\mathcal{H})$, and an ugly but straightforward induction on m shows that each $\delta^k(f(a_1, \dots, a_n))$ and $\delta^k(d(f(a_1, \dots, a_n)))$ lies in its domain, using the convergence of $\int |\hat{f}(s_1, \dots, s_n)|\, |p(s_1, \dots, s_n)|\, d^n s$ for p a polynomial of degree $\le m + 1$. Thus, $f(a_1, \dots, a_n) \in A_m$. Since $\mathcal{A}_\delta = \bigcap_{m \ge 1} A_m$, we conclude that $f(a_1, \dots, a_n) \in \mathcal{A}_\delta$. □

The C^∞ functional calculus is useful for constructing specific elements in the algebra.

5.2 Summability for spectral triples

5.2.1 Finite summability

Just as for Fredholm modules, we require a notion of summability for spectral triples. As for Fredholm modules, this is needed to write down explicit formulae for index pairings, Chern characters etc etc etc.

Definition 5.9. *A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is called finitely summable if there is some $s_0 > 0$ such that*

$$\mathrm{Trace}((1+\mathcal{D}^2)^{-s_0/2}) < \infty.$$

This is then true for all $s > s_0$ and we call

$$p = \inf\{s \in \mathbb{R}_+ : \mathrm{Trace}((1+\mathcal{D}^2)^{-s/2}) < \infty\}$$

the spectral dimension.

Remark What finitely summable means for a spectral triple with $\mathcal{A}$ nonunital and $(1+\mathcal{D}^2)^{-1/2}$ not a compact operator is still an open question, but see [GGISV, R2, R3].

Not all algebras have finitely summable spectral triples, even when they have finitely summable Fredholm modules (more on this later). We quote the following necessary condition due to Connes.

Theorem 5.10 (Connes, [C2])**.** *Let A be a unital C^*-algebra and $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ a finitely summable QC^1 spectral triple, with $\mathcal{A} \subset A$ dense. Then there exists a positive trace τ on A with $\tau(1) = 1$.*

So algebras with no normalised trace, like the Cuntz algebra, do not have finitely summable spectral triples associated to them.

Proposition 5.11. *If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a finitely summable QC^1 spectral triple with spectral dimension $p \geq 0$, then $(\mathcal{H}, F_{\mathcal{D}})$ is a $[p]+1$-summable Fredholm module for $\mathcal{A}$, where $[p]$ is the largest integer less than or equal to p.*

Proof. Let $a \in \mathcal{A}$ and recall

$$[F_{\mathcal{D}}, a] = [\mathcal{D}, a](1+\mathcal{D}^2)^{-1/2} - F_{\mathcal{D}}[(1+\mathcal{D}^2)^{1/2}, a](1+\mathcal{D}^2)^{-1/2} =: T(1+\mathcal{D}^2)^{-1/2}.$$

Now observe that T is bounded, and we want to show

$$T(1+\mathcal{D}^2)^{-1/2}T(1+\mathcal{D}^2)^{-1/2} \cdots T(1+\mathcal{D}^2)^{-1/2} \in \mathcal{L}^1(\mathcal{H})$$

where we have a product of $[p]+1$ terms. For each $\epsilon > 0$ we have $T(1+\mathcal{D}^2)^{-1/2} \in \mathcal{L}^{p+\epsilon}(\mathcal{H})$. As $[p] \leq p < [p]+1$, we can choose ϵ between p and $[p]+1$, and so $T(1+\mathcal{D}^2)^{-1/2} \in \mathcal{L}^{[p]+1}(\mathcal{H})$, the product is in $\mathcal{L}^1(\mathcal{H})$ and we are done. □

Remark Again using [CP1] we can replace QC^1 by QC^0.

The finitely summable Fredholm module we wind up with is not normalised in general. To obtain a normalised finitely summable Fredholm module, we follow the same recipe that we applied to the Hodge-de Rham example.

Lemma 5.12. *Let $(\mathcal{A},\mathcal{H},\mathcal{D})$ be a spectral triple. For any $m>0$ we define the double of $(\mathcal{A},\mathcal{H},\mathcal{D})$ to be the spectral triple $(\mathcal{A},\mathcal{H}_2,\mathcal{D}_m)$ with*

$$\mathcal{H}_2=\mathcal{H}\oplus\mathcal{H},\quad \mathcal{D}_m=\begin{pmatrix}\mathcal{D} & m\\ m & -\mathcal{D}\end{pmatrix}, a\to\begin{pmatrix} a & 0\\ 0 & 0\end{pmatrix}.$$

If $(\mathcal{A},\mathcal{H},\mathcal{D})$ is graded by γ, the double is graded by $\gamma\oplus-\gamma$. If $(\mathcal{A},\mathcal{H},\mathcal{D})$ is QC^k, $k=0,1,\ldots,\infty$, so is the double. If $(\mathcal{A},\mathcal{H},\mathcal{D})$ is finitely summable with spectral dimension p, the double is finitely summable with spectral dimension p. Moreover, the K-homology classes of $(\mathcal{A},\mathcal{H},\mathcal{D})$ and its double coincide for any $m>0$. This class can be represented by the normalised Fredholm module $(\mathcal{H}_2,\mathcal{D}_m|\mathcal{D}_m|^{-1})$.

Remark Most of this is folklore, and easy to prove. The explicit identification of the K-homology classes and the normalised representative can be found in [CPRS1].

We also want to know that when we pair with elements of K-theory we wind up with unbounded Fredholm operators whose index is independent of which 'Sobolev' version of the operator we choose to work with.

This does indeed work, as is shown in [H]. This relies on the smoothness of the spectral triple, the ability to choose smooth representatives of K-theory classes, and the pseudodifferential calculus for spectral triples. The latter is used to show that the abstract analogue of the elliptic estimates on manifolds are true. See [H] for an excellent discussion.

5.2.2 (n,∞)-summability and the Dixmier trace

Recall Weyl's theorem.

Theorem 5.13 (Weyl's theorem)**.** *Let P be an order d elliptic differential operator on a compact oriented manifold M of dimension n. Let $\{\lambda_k\}$ denote the eigenvalues of P ordered so $|\lambda_1|\leq|\lambda_2|\leq\cdots$ and repeated according to multiplicity. Then*

$$|\lambda_k|\sim Ck^{d/n}.$$

The constant C can also be computed, but we will leave that for a little while. First we will introduce some analytic machinery.

Let $\mathcal{H}$ be a separable Hilbert space. If $T\in\mathcal{K}(\mathcal{H})$, let $\mu_n(T)$ denote the n-th singular number of T; that is $\mu_n(T)$ is the n-th eigenvalue of $\sqrt{T^*T}$ when they are listed in nonincreasing order and repeated according to multiplicity. Let

$$\sigma_N(T)=\sum_{k=1}^{N}\mu_k(T)$$

be the N-th partial sum of the singular values.

For $p>1$ let

$$\mathcal{L}^{(p,\infty)}(\mathcal{H})=\{T\in\mathcal{K}(\mathcal{H}):\sigma_N(T)=O(N^{1-1/p})\}$$

and for $p=1$

$$\mathcal{L}^{(1,\infty)}(\mathcal{H})=\{T\in\mathcal{K}(\mathcal{H}):\sigma_N(T)=O(\log N)\}.$$

We will be mostly interested in $\mathcal{L}^{(1,\infty)}(\mathcal{H})$, however the following is useful: If $T_1, \ldots, T_m$ are in $\mathcal{L}^{(p_1,\infty)}, \ldots, \mathcal{L}^{(p_m,\infty)}$ respectively, and $1/p_1 + \cdots + 1/p_m = 1$ then $T_1 T_2 \cdots T_m \in \mathcal{L}^{(1,\infty)}$.

While the Schatten classes play a similar role to the L^p spaces of classical analysis, the $\mathcal{L}^{(p,\infty)}$ spaces play a role similar to weak L^p spaces.

What we would like to do is construct a functional on $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ by defining for $T \geq 0$

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{k=1}^{N} \mu_k(T).$$

However this formula need not define a linear functional, and it may not converge. The trick is to consider the sequence

$$(\frac{\sigma_2(T)}{\log 2}, \frac{\sigma_3(T)}{\log 3}, \frac{\sigma_4(T)}{\log 4}, ...)$$

and observe that this sequence is bounded. If it always converged, the limit would provide a linear functional which is a trace.

Unfortunately it does not always converge, and one must consider certain generalised limits which give a linear functional. We will denote by $\lim_\omega$ any such generalised limit, and observe that there are uncountably many such limits. For a fuller discussion see [CPS2, C1].

For any such choice the following is true.

Proposition 5.14. *For $T \geq 0$, $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$ define*

$$\mathrm{Tr}_\omega(T) = \lim_\omega \frac{1}{\log N} \sum_{k=1}^{N} \mu_k(T).$$

Then

1) $\mathrm{Tr}_\omega(T_1 + T_2) = \mathrm{Tr}_\omega(T_1) + \mathrm{Tr}_\omega(T_2)$*, so we can extend it by linearity to all of* $\mathcal{L}^{(1,\infty)}(\mathcal{H})$

2) If $T \geq 0$ then $\mathrm{Tr}_\omega(T) \geq 0$

3) If $S \in \mathcal{B}(\mathcal{H})$ and $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$ then $\mathrm{Tr}_\omega(ST) = \mathrm{Tr}_\omega(TS)$

Moreover for any trace class operator T we have $\mathrm{Tr}_\omega(T) = 0$

All this is true for any choice of ω, but in practise the value of the Dixmier trace on interesting operators is independent of the choice of ω: we call such operators measurable.

Here is a key criteria for measurability. First, for $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$, $T \geq 0$, define for $Re(s) > 1$

$$\zeta_T(s) = \mathrm{Trace}(T^s) = \sum_{k=1}^{\infty} \mu_k(T)^s.$$

Then

Proposition 5.15. *With $T \geq 0$ as above the following are equivalent:*

1) $(s-1)\zeta_T(s) \to L$ *as* $s \searrow 1$*;*

2) $\frac{1}{\log N}\sum_{k=1}^{N}\mu_k(T) \to L$ *as* $N \to \infty$*.*

In this case, the residue at $s = 1$ of $\zeta_T(s)$ is precisely $\mathrm{Tr}_\omega(T)$ and so the Dixmier trace of T is independent of ω.

In fact, one has to work extremely hard to come up with nonmeasurable operators, see [GVF] for an example, and 'naturally occurring' operators are measurable in all known cases. As a typical example, we quote the following from Connes.

Proposition 5.16. *Let M be an n dimensional compact manifold and let T be a classical pseudo-differential operator of order $-n$ (think of $T = (1+\mathcal{D}^2)^{-n/2}$ where $\mathcal{D}$ is order 1) acting on sections of a complex vector bundle $E \to M$. Then*

1) The corresponding operator T on $L^2(M, E)$ belongs to the ideal $\mathcal{L}^{(1,\infty)}$

2) The Dixmier trace $Tr_\omega(T)$ is independent of ω and is equal to the Wodzicki residue:

$$WRes(T) = \frac{1}{n(2\pi)^n}\int_{S^*M} trace_E(\sigma_T(x,\xi))dvol.$$

Here S^*M is the cosphere bundle, $\{\xi \in T^*M : \|\xi\|^2 = g^{\mu\nu}\xi_\mu\xi_\nu = 1\}$.

The amazing thing about the Wodzicki residue is it extends to a trace (the unique such trace) on the whole algebra of pseudodifferential operators of any order. This extension is simply to take the $-n$-th part of the symbol and integrate it over the cosphere bundle.

An important lesson is that the residue of the zeta function can be computed geometrically.

In the following we restrict attention to operators 'of Dirac type', by which we mean that the principal symbol of $\mathcal{D}$ is Clifford multiplication. This means that the symbol of $\mathcal{D}^2$ is given by $\sigma_{\mathcal{D}^2}(x,\xi) = \|\xi\|^2$.

Corollary 5.17. *Let $f \in C^\infty(M)$ and $\mathcal{D}$ be a first order self-adjoint elliptic operator 'of Dirac type' on the vector bundle E. Then the operator $f(1+\mathcal{D}^2)^{-n/2}$ acting on $L^2(E)$ is measurable and*

$$\mathrm{Tr}_\omega(f(1+\mathcal{D}^2)^{-n/2}) = \frac{rank(E)Vol(S^{n-1})}{n(2\pi)^n}\int_M f dvol.$$

Hence the representation of functions as multiplication operators, along with the spectrum of $\mathcal{D}$, is enough to recover the integral on a manifold using the Dixmier trace. This has stimulated interest in other spectral triples satisfying the following summability hypothesis.

Definition 5.18. *A spectral triple $(\mathcal{A}, \mathcal{H}\mathcal{D})$ is (n,∞)-summable if*

$$(1+\mathcal{D}^2)^{-n/2} \in \mathcal{L}^{(1,\infty)}(\mathcal{H}).$$

This definition is definitely in the context of unital algebras $\mathcal{A}$. For an approach to this definition when $\mathcal{A}$ is nonunital see [GGISV, R2, R3].

Observe that if a spectral triple is (n,∞)-summable, then the associated Fredholm module is $n+1$ summable. Also, the spectral dimension of such a triple is n.

Example 18. Examining the eigenvalues of the 'Dirac' operator for the noncommutative torus in Section 3.5 (for simplicity set $\tau = i$), we see that the eigenvalues obey Weyl's Theorem. This is not surprising since $\mathcal{D}$ and $\mathcal{H}$ are actually the same as in the commutative case. Hence the spectral triple for the noncommutative torus is $(2,\infty)$-summable with spectral dimension $p = 2$.

Exercise Prove the $(2,\infty)$-summability, and compute the Dixmier trace of $(1+\mathcal{D}^2)^{-1}$. *Hint* See [GVF].

Example 19. For the Cantor set spectral triple introduced in Example 7 we can also work out what is happenning.

If the gap between e_- and e_+ appears at the n-th stage of our construction (counting the interval $[0,1]$ as the 0-th stage), then $e_+ - e_- = 3^{-n}$. How many gaps are there with this length? Well, 2^{n-1} (except $n=0$). So the trace of $|\mathcal{D}|^{-s}$ for $s >> 1$ is

$$\zeta(s) = \sum_{n=0}^{\infty} 2^n 3^{-ns} = \frac{1}{1 - 2/3^s}.$$

This is finite for $s > \frac{\log 2}{\log 3}$ and this formula provides a meromorphic continuation of $\zeta(s)$ whose only singularities are simple poles at $s = (\log 2 + 2k\pi i)/\log 3$. The number $\log 2/\log 3$ is the Hausdorff dimension of the Cantor set.

Exercise What is the reside at $s = \log 2/\log 3$?

The relationship between the Dixmier trace and the zeta function is well described in [CPS2], and the definitive results are in [CRSS].

5.2.3 θ-summability

Definition 5.19. *A spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D})$ is θ-summable if for all $t > 0$ we have*

$$\text{Trace}(e^{-t\mathcal{D}^2}) < \infty.$$

Example 20. Sadly, not all interesting spectral triples are finitely summable, so the local index formula is not available. The examples arising from supersymmetric quantum field theory are generically not finitely summable, but rather than take a detour into physics, we will look at examples coming from group C^*-algebras.

All of this material, plus the construction of the metric on the state space first appeared in the beautiful paper [C2].

Let Γ be a finitely generated group, and let $\mathbb{C}\Gamma$ denote the group ring of Γ. Then the C^*-algebra $C^*_{red}(\Gamma)$, called the reduced C^*-algebra of Γ, is the norm closure of the action of $\mathbb{C}\Gamma$ acting on $l^2(\Gamma)$ in the left regular representation. For $\psi \in l^2(\Gamma)$ and $g \in \Gamma$, the left regular representation is given by

$$(\lambda(g)\psi)(k) = \psi(g^{-1}k). \tag{5.4}$$

Define a length function on Γ to be a function $L : \Gamma \to \mathbb{R}^+$ such that

1) $L(gh) \leq L(g) + L(h)$ for all $g, h \in \Gamma$

2) $L(g^{-1}) = L(g)$ for all $g \in \Gamma$

3) $L(1) = 0$

The prototypical example is the word length function. Let $G \subset \Gamma$ be a generating set. Then for all $g \in \Gamma$, $g = g_1 \cdots g_n$ for some n where $g_i \in G$ for all $i = 1, ..., n$. This expression is not unique, but can be made so by removing terms like $g_i g_i^{-1}$ and other terms that cancel owing to the relations amongst group elements. The resulting expression is called the reduced word of g. If the reduced word of $g \in \Gamma$ is $g = g_1 \cdots g_n$ then the word length function of g is simply n.

Using length functions we can construct spectral triples.

Lemma 5.20. *Let Γ be a discrete group and L a length function on Γ. Let $\mathcal{D}$ be the operator of multiplication by L on $\mathcal{H} = l^2(\Gamma)$. If $L(g) \to \infty$ as $g \to \infty$ then*

1) $(\mathbb{C}(\Gamma), \mathcal{H}, \mathcal{D})$ is a spectral triple.

2) $\| [\mathcal{D}, \lambda(g)] \| = L(g)$ for all $g \in \Gamma$.

Proof. To show that for a dense subalgebra $\mathcal{A} \subset C^*(\Gamma)$ the commutators $[\mathcal{D}, a]$ are bounded, it suffices to show that for all $g \in \Gamma$, the commutator $[\mathcal{D}, \lambda(g)]$ is bounded (the group ring $\mathbb{C}\Gamma$ is dense). We compute

$$\begin{aligned}(\mathcal{D}\lambda(g)\psi)(k) - (\lambda(g)\mathcal{D}\psi)(k) &= \mathcal{D}\psi(g^{-1}k) - \lambda(g)L(k)\psi(k) \\ &= L(g^{-1}k)\psi(g^{-1}k) - L(k)\psi(g^{-1}k). \end{aligned} \tag{5.5}$$

However

$$|L(g^{-1}k) - L(k)| \leq |L(g^{-1}) + L(k) - L(k)| = L(g), \tag{5.6}$$

so this is bounded.

Now for any real number $x \in \mathbb{R}$, let $K_x \subset \Gamma$ be those group elements with $L(g) = x$. Let ψ_x be the function in $l^2(\Gamma)$ with $\psi_x \equiv 1$ on K_x and zero elsewhere. Then

$$\mathcal{D}\psi_x = x\psi_x. \tag{5.7}$$

As Γ is discrete, L takes on only a discrete number of values. Thus there are a countable number of ψ_xs and corresponding eigenvalues x. With the assumption that $L(g) \to \infty$ as $g \to \infty$, we see that $\mathcal{D}$ is unbounded, has countably many eigenvalues of finite multiplicity, and this is enough to conclude that $\mathcal{D}$ has compact resolvent. This shows that we have a spectral triple. We do not know if it is even or odd.

Lastly, let ψ_1 be the function which is 1 on $1 \in \Gamma$ and zero elsewhere. Then

$$\begin{aligned}([\mathcal{D}, \lambda(g)]\psi_1)(k) &= (\mathcal{D}\lambda(g)\psi_1)(k) - (\lambda(g)\mathcal{D}\psi_1)(k)\\ &= (\mathcal{D}\psi_1)(g^{-1}k) - (\lambda(g)L(k)\psi_1)(k)\\ &= (L(g^{-1}k) - L(k))\psi_1(g^{-1}k)\\ &= \begin{cases} 0 & k \neq g^{-1}\\ -L(k) & k = g^{-1}\end{cases}\\ &= -L(g)\delta_{k,g^{-1}}. \end{aligned} \tag{5.8}$$

So as we showed that $\| [\mathcal{D}, \lambda(g)] \| \leq L(g)$, the above calculation shows that equality always holds, proving the second assertion of the lemma. □

So we have a spectral triple, a priori it is odd (ungraded). We are interested in seeing whether it is finitely summable.

Theorem 5.21 (Connes, [C2]). *Let Γ be a discrete group containing the free group on two generators. Let $\mathcal{H}$ be any representation of $C^*(\Gamma)$, absolutely continuous with respect to the canonical trace on $C^*(\Gamma)$. Then there does not exist a self-adjoint operator $\mathcal{D}$ on $\mathcal{H}$ such that $(\mathcal{H}, \mathcal{D})$ is a finitely summable spectral triple for $C^*_{red}(\Gamma)$.*

Remark There may be finitely summable Fredholm modules for such a group algebra. In particular, one is known for the free group on 2 generators. The culprit here is the lack of hyperfiniteness of the group von Neumann algebra.

Theorem 5.22 (Connes, [C2]). *If Γ is an infinite discrete group with property T, then there exists no finitely summable spectral triple for $C^*_{red}(\Gamma)$.*

Again, there are interesting finitely summable Fredholm modules for such groups.

Have we run into a fundamental problem in finding geometric representatives for K-homology classes?

Theorem 5.23 (Connes, [C2]). *Let Γ be a finitely generated discrete group, and l the word length function, relative to some generating subset. Let $\mathcal{H} = l^2(\Gamma)$, with $C^*_{red}(\Gamma)$ acting by multiplication and let $\mathcal{D}$ be multiplication by the word length function l. Then $(\mathcal{H}, \mathcal{D})$ is a θ-summable spectral triple for $C^*_{red}(\Gamma)$.*

So it would seem that the fundamental problem we ran into is that some group C^*-algebras are 'infinite dimensional noncommutative spaces'.

5.3 Analytic formulae for the index

There are analytic formulae which compute the pairing between a spectral triple and K-theory with no reference to cyclic cohomology. Nevertheless, it is via these formulae that the link to cyclic cohomology is made in [CPRS2, CPRS3].

Theorem 5.24 (McKean-Singer Formula). *Let $\mathcal{D}$ be an unbounded self-adjoint operator with compact resolvent. Let γ be a self-adjoint unitary which anticommutes with $\mathcal{D}$. Finally, let f be a continuous even function on* $\mathbf{R}$ *with $f(0) \neq 0$ and $f(\mathcal{D})$ trace-class. Let $\mathcal{D}^+ = P^\perp \mathcal{D} P$ where $P = (1+\gamma)/2$ and $P^\perp = 1 - P$. Then $D^+ : P\mathcal{H} \to P^\perp\mathcal{H}$ is Fredholm and*

$$\mathrm{Index}(D^+) = \frac{1}{f(0)} \mathrm{Trace}\left(\gamma f(D)\right). \tag{5.9}$$

This version (actually a stronger version valid in semfinite von Neumann algebras) can be found in [CPRS3]. The traditional function used in this context is $f(x) = e^{-tx^2}$, $t > 0$, so the formula becomes:

$$\mathrm{Index}(\mathcal{D}^+) = \mathrm{Trace}\left(\gamma e^{-t\mathcal{D}^2}\right).$$

The operator $e^{-t\mathcal{D}^2}$ is often called a **heat kernel**, being the solution of the 'heat equation' $\partial_t A(t) + \mathcal{D}^2 A(t) = 0$. However for a finitely summable spectral triple, functions such as $(1+\mathcal{D}^2)^{-s/2}$ for s large provide a natural alternative.

The McKean-Singer formula goes back to the early seventies (at least) and has been rediscovered and used by numerous people (I make no attempt at a proper attribution).

On the other hand, an analogous analytic formula for the odd pairing is totally new, [CP1, CP2].

Theorem 5.25 (Carey-Phillips Spectral Flow formula). *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a finitely summable spectral triple with spectral dimension $p \geq 1$. Let $u \in \mathcal{A}$ be unitary and let P be the spectral projection of $\mathcal{D}$ corresponding to the interval $[0, \infty)$. Then for any $s > p$*

$$\mathrm{Index}(PuP) = sf(\mathcal{D}, u\mathcal{D}u^*) = \frac{1}{C_{s/2}} \int_0^1 \tau(u[\mathcal{D}, u^*](1 + (\mathcal{D} + tu[\mathcal{D}, u^*])^2)^{-s/2})dt, \tag{5.10}$$

with $C_{s/2} = \int_{-\infty}^{\infty} (1+x^2)^{-s/2} dx$.

Both of the analytic formulae are scale invariant. By this we mean that if we replace $\mathcal{D}$ by $\epsilon\mathcal{D}$, for $\epsilon > 0$, in the right hand side of (5.10) or (5.9), then the left hand side is unchanged, since in both cases the index is invariant with respect to change of scale.

Rewriting the 'constant' $C_{s/2}$ as

$$C_{s/2} = \frac{\Gamma(s - 1/2)\Gamma(1/2)}{\Gamma(s)}$$

we see that in fact the integral formula in (5.10) can be given a meromorphic continuation (as a function of s) by setting

$$\mathrm{Index}(PuP) C_{s/2} = \int_{\mathbf{0}}^{\mathbf{1}} \tau(\mathbf{u}[\mathcal{D}, \mathbf{u}^*](\mathbf{1} + (\mathcal{D} + \mathbf{tu}[\mathcal{D}, \mathbf{u}^*])^{\mathbf{2}})^{-\mathbf{s/2}})\mathbf{dt}.$$

Here we have written the right hand side in bold face to indicate that we are thinking of the meromorphically continued function. Since the residue of $C_{s/2}$ at $s = 1/2$ is 1, we also have

$$\mathrm{Index}(PuP) = res_{s=1/2} \int_{\mathbf{0}}^{\mathbf{1}} \tau(\mathbf{u}[\mathcal{D}, \mathbf{u}^*](\mathbf{1} + (\mathcal{D} + \mathbf{tu}[\mathcal{D}, \mathbf{u}^*])^{\mathbf{2}})^{-\mathbf{s/2}})\mathbf{dt}.$$

This observation is the starting point for the proof of the local index theorem in [CPRS2]. A suitable choice of functions allows a similar analysis in the even case; we refer to [CPRS3].

The JLO-cocycle can also be derived from an analytic formula. In the even case, the McKean-Singer formula suffices, while in the odd case one uses the θ-summable spectral flow formula of Carey-Phillips: Let $\mathcal{D}_t = \mathcal{D} + tu[\mathcal{D}, u^*]$ for $u \in \mathcal{A}$ unitary, and then

$$\mathrm{Index}(PuP) = \frac{1}{\sqrt{\pi}} \int_0^1 \mathrm{Trace}(u[\mathcal{D}, u^*] e^{-\mathcal{D}_t^2}) dt.$$

The derivation of the JLO-cocycle from the analytic formula is in [CP2].

Apart from group C^*-algebras, the JLO cocycle has been used in supersymmetric quantum field theory and as a tool for studying the Chern character and its representatives. Indeed, the starting point for Connes and Moscovici's original proof of the local index theorem was the JLO cocycle.

5.4 Appendix: Pseudodifferential calculus for spectral triples

This section can be skipped on a first reading, and is really here only if you want some way to prove the elliptic estimate and Lemma 2.10 for spectral triples. The proofs of the results here can be found in [CPRS2].

In this section we introduce the terminology and basic results of the Connes-Moscovici pseudodifferential calculus, first introduced in [CM, C4]. This calculus works in great generality, only needing an unbounded self-adjoint operator D. The proofs of some of these results are quite tricky, and we have omitted them. The original proofs were simplified by Higson in [H], and these notes are from [CPRS2], where Higson's ideas were also utilised.

Just as we did in the remarks following Definition 5.1, we set

$$|D|_1 = (1 + D^2)^{1/2}, \quad \delta_1(T) = [|D|_1, T], \quad T \in \mathrm{dom}\delta.$$

We follow the discussion of the pseudodifferential calculus in [C4], using $|D|_1$ and δ_1, instead of $|D|$ and δ. In order to ensure that the calculus works in this modified setting we flesh out explanations in [C4] and record some elementary properties which are trivial to prove, but are often used without comment. The most important results are Proposition 5.30 and its Corollary 5.33.

So let $D : \mathrm{dom}D \subseteq \mathcal{H} \longrightarrow \mathcal{H}$ be an unbounded self-adjoint operator on the Hilbert space $\mathcal{H}$. For all $k \geq 0$, we set

$$\mathcal{H}_k = \mathrm{dom}(1 + D^2)^{k/2} = \mathrm{dom}|D|^k \subseteq \mathcal{H}$$

and $\mathcal{H}_\infty = \cap_{k \geq 0} \mathcal{H}_k$. Recall that the graph norm topology makes $\mathcal{H}_k$ into a Hilbert space with norm $\| \cdot \|_k$ given by

$$\| \xi \|_k^2 = \| \xi \|^2 + \| (1 + D^2)^{k/2} \xi \|^2$$

where $\| \cdot \|$ is the norm on $\mathcal{H}$.

We assume that all of our operators T, in particular D, preserve $\mathcal{H}_\infty$, so $T : \mathcal{H}_\infty \to \mathcal{H}_\infty$. In this way, all computations involving bounded or unbounded operators make sense on the dense subspace $\mathcal{H}_\infty$.

Definition 5.26. *For $r \in \mathbb{R}$, let op^r be the linear space of operators mapping $\mathcal{H}_\infty \to \mathcal{H}_\infty$ which are continuous in the norms $(\mathcal{H}_\infty, \|\cdot\|_k) \to (\mathcal{H}_\infty, \|\cdot\|_{k-r})$ for all k such that $k - r \geq 0$.*

Example 21. The operators $|D|^r$ and $(1 + D^2)^{r/2}$ are in op^r.

Lemma 5.27. *[compare Lemma 1.1 of [C4]] Let $b \in \cap_{n\geq 0}\mathrm{dom}\delta_1^n$. With $\sigma_1(b) = |D|_1 b|D|_1^{-1}$ and $\varepsilon_1(b) = \delta_1(b)|D|_1^{-1}$ we have*

1) $\sigma_1 = Id + \varepsilon_1$,

2) $\varepsilon_1^n(b) = \delta_1^n(b)|D|_1^{-n} \in \mathcal{N} \quad \forall n$,

3) $\sigma_1^n(b) = (Id + \varepsilon_1)^n(b) = \sum_{k=0}^n \binom{n}{k}\delta_1^k(b)|D|_1^{-k} \in \mathcal{N} \quad \forall n$.

Proof. The first statement is straightforward. The second follows because δ_1 is a derivation with $\delta_1(|D|_1) = 0$. The third is just the binomial theorem applied to 1). □

Similarly, if $b \in op^0$, $\sigma_1^{-n}(b) := |D|_1^{-n} b|D|_1^n \in \mathcal{N}$ for all n and

$$|D|_1^{-n} b|D|_1^n = \sum_{k=0}^n \binom{n}{k} |D|_1^{-k}\delta_1^k(b).$$

Corollary 5.28. *If $b \in \cap_{n\geq 0}\mathrm{dom}\delta_1^n$ then $b \in op^0$.*

Observe that by the above Lemma, if $b \in op^0$ then $b - \sigma_1(b) = -\varepsilon_1(b) = -\delta_1(b)|D|_1^{-1} \in op^{-1}$. Thus if $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a QC^∞ spectral triple, and $b = a$ or $[\mathcal{D}, a]$ for $a \in \mathcal{A}$, then (with $\mathcal{D}$ playing the role of D) $b \in op^0$ and $b - |\mathcal{D}|_1 b|\mathcal{D}|_1^{-1} \in op^{-1}$.

Example 22. In even the most elementary case $\mathcal{A} = C^\infty(S^1)$, $\mathcal{H} = L^2(S^1)$, $a = M_z$, the operator of multiplication by z, and $\mathcal{D} = \frac{1}{i}\frac{d}{d\theta}$ one can easily see that $a \in \cap_{n\geq 0}\mathrm{dom}\delta_1^n$ but that $[\mathcal{D}^2, a]$ is not bounded. In general, $[\mathcal{D}^2, a]$ is about the same size as $|\mathcal{D}|$.

Definition 5.29. *We define the commuting operators L_1, R_1 on the space of operators on $\mathcal{H}_\infty$ by*

$$L_1(T) = (1 + D^2)^{-1/2}[D^2, T] = |D|_1^{-1}[|D|_1^2, T],$$

$$R_1(T) = [D^2, T](1 + D^2)^{-1/2} = [|D|_1^2, T]|D|_1^{-1}.$$

Proposition 5.30. *[Compare Lemma 2 [C4]] For all $b \in op^0$ the following are equivalent:*

1) $b \in \bigcap_{n\geq 0}\mathrm{dom}\delta_1^n$,

2) $b \in \bigcap_{k,l\geq 0}\mathrm{dom}L_1^k \circ R_1^l$.

If $T : \mathcal{H}_\infty \to \mathcal{H}_\infty$ then it is useful to denote by $T^{(1)} := [D^2, T]$ and $T^{(k)} = [D^2, [D^2, \cdots [D^2, T]\cdots]]$ (k commutators).

Definition 5.31. *For $r \in \mathbb{R}$*

$$OP^r = |D|_1^r \left(\bigcap_{n \geq 0} \mathrm{dom}\delta_1^n \right) \subseteq op^r \cdot op^0 \subseteq op^r.$$

If $T \in OP^r$ we say that the order of T is (at most) r. The definition is actually symmetric, since for r an integer (at least) we have by Lemma 5.27

$$OP^r = |D|_1^r \left(\bigcap dom\delta_1^n \right) = |D|_1^r \left(\bigcap dom\delta_1^n \right) |D|_1^{-r}|D|_1^r \subseteq \left(\bigcap dom\delta_1^n \right) |D|_1^r.$$

From this we easily see that $OP^r \cdot OP^s \subseteq OP^{r+s}$. Finally, we note that if $b \in OP^r$ for $r \geq 0$, then since $b = |D|_1^r a$ for some $a \in OP^0$, we get $[|D|_1, b] = |D|_1^r[|D|_1, a] = |D|_1^r\delta_1(a)$, so $[|D|_1, b] \in OP^r$.

Remarks An operator $T \in OP^r$ if and only if $|D|_1^{-r}T \in \cap_{n\geq 0}\mathrm{dom}\delta_1^n$. Observe that operators of order at most zero are bounded. If $|D|_1^{-1}$ is p-summable and T has order $-n$ then, T is p/n-summable.

Important Observations
1) If f is a bounded Borel function then $f(D) \in \mathcal{B}(\mathcal{H})$ and $\delta_1(f(D)) = 0$, implies $f(D) \in OP^0$.
2) If g is an unbounded Borel function such that $1/g$ is bounded on $spec(D)$ and both $g(D)|D|_1^{-1}$ and $g(D)^{-1}|D|_1$ are bounded, then for each r, $OP^r = |g(D)|^r OP^0$. This follows since OP^0 is an algebra and both $|g(D)|^r|D|_1^{-r}$ and $g(D)^{-r}|D|_1^r$ are in OP^0. We note that if $|D|$ is **not** invertible then we get strict containment $|D|^r OP^0 \subset |D|_1^r OP^0$. These observations prove the next Lemma.

Lemma 5.32. *If $\mu \in \mathbb{C}$ is in the resolvent set of D^2 then*

$$OP^r = |(\mu - D^2)^{1/2}|^r \left(\bigcap_{n\geq 0} \mathrm{dom}\delta_1^n \right).$$

Corollary 5.33. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a QC^∞ spectral triple, and suppose $a \in \mathcal{A}$. Then for $n \geq 0$, $a^{(n)}$ and $[\mathcal{D}, a]^{(n)}$ are in OP^n.*

Next we describe the asymptotic expansions introduced by Connes and Moscovici in [C4, CM]. Their principal result is that if $T \in OP^k$ for k integral, then for any $z \in \mathbb{C}$

$$\begin{aligned}(1 + D^2)^z T \;&=\; T(1 + D^2)^z + zT^{(1)}(1 + D^2)^{z-1} + \frac{z(z-1)}{2}T^{(2)}(1 + D^2)^{z-2} + \cdots \\ &\cdots + \frac{z(z-1)\cdots(z-n+1)}{n!}T^{(n)}(1 + D^2)^{z-n} + P,\end{aligned}$$

where $P \in OP^{k-(n+1)+2Re(z)}$. This result is proved in both of the papers [C4, CM], but subsequently a simpler proof has been given by Higson [H].

We briefly sketch the idea behind Higson's proof. So we suppose that we have a QC^∞ spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ with dimension $p \geq 1$. We use $\mathcal{D}$ to define the pseudodifferential calculus on $\mathcal{H}$ as in the previous section. Let $Q = (1 + s^2 + D^2)$ where D is $\tilde{\mathcal{D}}$ as defined in Section 5.3 and where $s \in [0, \infty)$. For $Re(z) > p/2$ we write Q^{-z} using Cauchy's formula

$$Q^{-z} = \frac{1}{2\pi i} \int_l \lambda^{-z}(\lambda - Q)^{-1} d\lambda,$$

where l is a vertical line $\lambda = a + iv$ parametrized by $v \in \mathbb{R}$ with $0 < a < 1/2$ fixed. One checks that the integral indeed converges in operator norm and by using the spectral theorem for Q (in terms of its spectral resolution) it converges to Q^{-z} (principal branch). Computing commutators of Q^{-z} with an operator $T \in OP^m$ then reduces to an iterative calculation of commutators with $(\lambda - Q)^{-1}$. The exact result we need is the following.

Lemma 5.34. *Let m, n, k be non-negative integers and $T \in OP^m$. Then*

$$\begin{aligned}(\lambda - Q)^{-n}T &= T(\lambda - Q)^{-n} + nT^{(1)}(\lambda - Q)^{-(n+1)} + \frac{n(n+1)}{2}T^{(2)}(\lambda - Q)^{-(n+2)} + \cdots \\ \cdots &+ \binom{n+k-1}{k} T^{(k)}(\lambda - Q)^{-(n+k)} + P(\lambda) \\ &= \sum_{j=0}^{k} \binom{n+j-1}{j} T^{(j)}(\lambda - Q)^{-(n+j)} + P(\lambda)\end{aligned}$$

where the remainder $P(\lambda)$ has order $-(2n + k - m + 1)$ and is given by

$$P(\lambda) = \sum_{j=1}^{n} \binom{j+k-1}{k} (\lambda - Q)^{j-n-1} T^{(k+1)} (\lambda - Q)^{-j-k}.$$

Corollary. *Let n, M be positive integers and $A \in OP^k$. Let $R = (\lambda - Q)^{-1}$. Then,*

$$R^n A R^{-n} = \sum_{j=0}^{M} \binom{n+j-1}{j} A^{(j)} R^j + P$$

where

$$P = \sum_{j=1}^{n} \binom{j+M-1}{M} R^{n+1-j} A^{(M+1)} R^{M+j-n}$$

and P has order $k - M - 1$.

Chapter 6

The Chern character of spectral triples

6.1 The local index formula for finitely summable smooth spectral triples

What is the analogue of the Atiyah-Singer local index theorem in noncommutative geometry? This question was answered by Connes and Moscovici in the paper [CM]. Improved statements and proofs have been given in [H, CPRS2, CPRS3].

The (finite) summability conditions give a half-plane where the function

$$z \mapsto \tau((1+\mathcal{D}^2)^{-z}) \tag{6.1}$$

is well-defined and holomorphic. In [C4, CM], a stronger condition was imposed in order to prove the local index formula. This condition not only specifies a half-plane where the function in (6.1) is holomorphic, but also that this function analytically continues to $\mathbb{C}$ minus some discrete set. We clarify this in the following definitions.

Definition 6.1. *Let $(\mathcal{A},\mathcal{H},\mathcal{D})$ be a QC^∞ spectral triple. The algebra $\mathcal{B}(\mathcal{A}) \subseteq \mathcal{N}$ is the algebra of polynomials generated by $\delta^n(a)$ and $\delta^n([\mathcal{D},a])$ for $a \in \mathcal{A}$ and $n \geq 0$. A QC^∞ spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D})$ has* **discrete dimension spectrum** *$Sd \subseteq \mathbb{C}$ if Sd is a discrete set and for all $b \in \mathcal{B}(\mathcal{A})$ the function $\tau(b(1+\mathcal{D}^2)^{-z})$ is defined and holomorphic for $Re(z)$ large, and analytically continues to $\mathbb{C} \setminus Sd$. We say the dimension spectrum is* **simple** *if this zeta function has poles of order at most one for all $b \in \mathcal{B}(\mathcal{A})$,* **finite** *if there is a $k \in \mathbb{N}$ such that the function has poles of order at most k for all $b \in \mathcal{B}(\mathcal{A})$ and* **infinite***, if it is not finite.*

Connes and Moscovici impose the discrete dimension spectrum assumption to prove their original version of the local index formula.

The dimension spectrum idea is quite attractive in a number of respects. The dimension spectrum of a direct sum of spectral triples is the union of the dimension spectra of the summands. The dimension spectrum of a product consists of sums of elements in the dimension spectra of the 'prodands'.

New proofs of the local index formula were presented by Nigel Higson, and by Carey, Phillips, Rennie, Sukochev. These were much simpler, more widely applicable and in the case of [CPRS1, CPRS2], required much less restriction on the zeta functions, and in particular did not require the discrete dimension spectrum hypothesis.

We will introduce some notation and definitions and then state the local index formula.

Denote multi-indices by $(k_1, ..., k_m)$, $k_i = 0, 1, 2, ...$, whose length m will always be clear from the context and let $|k| = k_1 + \cdots + k_m$. Define

$$\alpha(k) = \frac{k_1!k_2!\cdots k_m!}{(k_1+1)(k_1+k_2+2)\cdots(|k|+m)}$$

and the numbers $\tilde{\sigma}_{n,j}$ and $\sigma_{n,j}$ are defined by the equalities

$$\prod_{j=0}^{n-1}(z+j+1/2) = \sum_{j=0}^{n} z^j \tilde{\sigma}_{n,j}, \qquad \text{and} \qquad \prod_{j=0}^{n-1}(z+j) = \sum_{j=1}^{n} \sigma_{n,j}.$$

If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a QC^∞ spectral triple and $T \in \mathcal{N}$ then $T^{(n)}$ is the n^{th} iterated commutator with $\mathcal{D}^2$, that is, $[\mathcal{D}^2, [\mathcal{D}^2, [\cdots, [\mathcal{D}^2, T]\cdots]]]$.

Definition 6.2. *If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a QC^∞ spectral triple, we call*

$$q = \inf\{k \in \mathbb{R} : \tau((1+\mathcal{D}^2)^{-k/2}) < \infty\}$$

the spectral dimension of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. We say that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ has isolated spectral dimension if for b of the form

$$b = a_0[\mathcal{D}, a_1]^{(k_1)} \cdots [\mathcal{D}, a_m]^{(k_m)}(1+\mathcal{D}^2)^{-m/2-|k|}$$

the zeta functions

$$\zeta_b(z-(1-q)/2) = \mathrm{Trace}(b(1+\mathcal{D}^2)^{-z+(1-q)/2})$$

have analytic continuations to a deleted neighbourhood of $z = (1-q)/2$.

Remark Observe that we allow the possibility that the analytic continuations of these zeta functions may have an essential singularity at $z = (1-q)/2$. All that is necessary for us is that the residues at this point exist. Note that discrete dimension spectrum implies isolated spectral dimension.

Now we define, for $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ having isolated spectral dimension and

$$b = a_0[\mathcal{D}, a_1]^{(k_1)} \cdots [\mathcal{D}, a_m]^{(k_m)}(1+\mathcal{D}^2)^{-m/2-|k|}$$

$$\tau_j(b) = res_{z=(1-q)/2}(z-(1-q)/2)^j \zeta_b(z-(1-q)/2).$$

The hypothesis of isolated spectral dimension is clearly necessary here in order to define the residues. Let Q be the spectral projection of $\mathcal{D}$ corresponding to the interval $[0, \infty)$

In [CPRS2] we proved the following result:

Theorem 6.3 (Odd local index formula)**.** *Let $(\mathcal{A},\mathcal{H},\mathcal{D})$ be an odd finitely summable QC^∞ spectral triple with spectral dimension $q \geq 1$. Let $N = [q/2]+1$ where $[\cdot]$ denotes the integer part, and let $u \in \mathcal{A}$ be unitary. Then*

1) $$index(QuQ) = \frac{1}{\sqrt{2\pi i}} res_{r=(1-q)/2}\left(\sum_{m=1,odd}^{2N-1}\phi_m^r(Ch_m(u))\right)$$

where for $a_0,...,a_m \in \mathcal{A}$, $l = \{a+iv : v \in \mathbb{R}\}$, $0 < a < 1/2$, $R_s(\lambda) = (\lambda-(1+s^2+\mathcal{D}^2))^{-1}$ and $r > 0$ we define $\phi_m^r(a_0,a_1,...,a_m)$ to be

$$\frac{-2\sqrt{2\pi i}}{\Gamma((m+1)/2)}\int_0^\infty s^m \,\mathrm{Trace}\left(\frac{1}{2\pi i}\int_l \lambda^{-q/2-r}a_0R_s(\lambda)[\mathcal{D},a_1]R_s(\lambda)\cdots[\mathcal{D},a_m]R_s(\lambda)d\lambda\right)ds.$$

In particular the sum on the right hand side of 1) *analytically continues to a deleted neighbourhood of $r=(1-q)/2$ with* at worst *a simple pole at $r = (1-q)/2$. Moreover, the complex function-valued cochain $(\phi_m^r)_{m=1,odd}^{2N-1}$ is a (b,B) cocycle for $\mathcal{A}$ modulo functions holomorphic in a half-plane containing $r = (1-q)/2$.*

2) The index is also the residue of a sum of zeta functions:

$$\frac{1}{\sqrt{2\pi i}}res_{r=(1-q)/2}\Bigg(\sum_{m=1,odd}^{2N-1}\sum_{|k|=0}^{2N-1-m}\sum_{j=0}^{|k|+(m-1)/2}(-1)^{|k|+m}\alpha(k)\Gamma((m+1)/2)\tilde{\sigma}_{|k|+(m-1)/2,j}$$
$$(r-(1-q)/2)^j\,\mathrm{Trace}\left(u^*[\mathcal{D},u]^{(k_1)}[\mathcal{D},u^*]^{(k_2)}\cdots[\mathcal{D},u]^{(k_m)}(1+\mathcal{D}^2)^{-m/2-|k|-r+(1-q)/2}\right)\Bigg).$$

In particular the sum of zeta functions on the right hand side analytically continues to a deleted neighbourhood of $r = (1-q)/2$ and has at worst *a simple pole at $r = (1-q)/2$.*

3) If $(\mathcal{A},\mathcal{H},\mathcal{D})$ also has isolated spectral dimension then

$$index(QuQ) = \frac{1}{\sqrt{2\pi i}}\sum_m \phi_m(Ch_m(u))$$

where for $a_0,...,a_m \in \mathcal{A}$

$$\begin{aligned}\phi_m(a_0,...,a_m) &= res_{r=(1-q)/2}\phi_m^r(a_0,...,a_m) = \sqrt{2\pi i}\sum_{|k|=0}^{2N-1-m}(-1)^{|k|}\alpha(k)\times\\ &\times \sum_{j=0}^{|k|+(m-1)/2}\tilde{\sigma}_{(|k|+(m-1)/2),j}\tau_j\left(a_0[\mathcal{D},a_1]^{(k_1)}\cdots[\mathcal{D},a_m]^{(k_m)}(1+\mathcal{D}^2)^{-|k|-m/2}\right),\end{aligned}$$

and $(\phi_m)_{m=1,odd}^{2N-1}$ is a (b,B) cocycle for $\mathcal{A}$. When $[q]=2n$ is even, the term with $m = 2N-1$ is zero, and for $m = 1,3,...,2N-3$, all the top terms with $|k| = 2N-1-m$ are zero.

Corollary 6.4. *For $1 \leq p < 2$, the statements in* 3) *of Theorem 6.3 are true without the assumption of isolated dimension spectrum.*

For even spectral triples we have

Theorem 6.5 (Even local index formula)**.** *Let $(\mathcal{A},\mathcal{H},\mathcal{D})$ be an even QC^∞ spectral triple with spectral dimension $q\geq 1$. Let $N=[\frac{q+1}{2}]$, where $[\cdot]$ denotes the integer part, and let $p\in\mathcal{A}$ be a self-adjoint projection. Then*

1) $$Ind(p\mathcal{D}^+p)=res_{r=(1-q)/2}\left(\sum_{m=0,even}^{2N}\phi_m^r(Ch_m(p))\right)$$
where for $a_0,...,a_m\in\mathcal{A}$, $l=\{a+iv : v\in\mathbb{R}\}$, $0<a<1/2$, $R_s(\lambda)=(\lambda-(1+s^2+\mathcal{D}^2))^{-1}$ and $r>1/2$ we define $\phi_m^r(a_0,a_1,...,a_m)$ to be

$$\frac{(m/2)!}{m!}\int_0^\infty 2^{m+1}s^m\,Trace\left(\gamma\frac{1}{2\pi i}\int_l\lambda^{-q/2-r}a_0R_s(\lambda)[\mathcal{D},a_1]R_s(\lambda)\cdots[\mathcal{D},a_m]R_s(\lambda)d\lambda\right)ds.$$

In particular the sum on the right hand side of 1) *analytically continues to a deleted neighbourhood of $r=(1-q)/2$ with* at worst *a simple pole at $r=(1-q)/2$. Moreover, the complex function-valued cochain $(\phi_m^r)_{m=0,even}^{2N}$ is a (b,B) cocycle for $\mathcal{A}$ modulo functions holomorphic in a half-plane containing $r=(1-q)/2$.*

2) The index, $Ind(p\mathcal{D}^+p)$ is also the residue of a sum of zeta functions:

$$res_{r=(1-q)/2}\Bigg(\sum_{m=0,even}^{2N}\sum_{|k|=0}^{2N-m}\sum_{j=1}^{|k|+m/2}(-1)^{|k|+m/2}\alpha(k)\frac{(m/2)!}{2m!}\sigma_{|k|+m/2,j}\times$$
$$\times(r-(1-q)/2)^j\,Trace\left(\gamma(2p-1)[\mathcal{D},p]^{(k_1)}[\mathcal{D},p]^{(k_2)}\cdots[\mathcal{D},p]^{(k_m)}(1+\mathcal{D}^2)^{-m/2-|k|-r+(1-q)/2}\right)\Bigg),$$

(for $m=0$ we replace $(2p-1)$ by $2p$). In particular the sum of zeta functions on the right hand side analytically continues to a deleted neighbourhood of $r=(1-q)/2$ and has at worst *a simple pole at $r=(1-q)/2$.*

3) If $(\mathcal{A},\mathcal{H},\mathcal{D})$ also has isolated spectral dimension then

$$Ind(p\mathcal{D}^+p)=\sum_{m=0,even}^{2N}\phi_m(Ch_m(p))$$

where for $a_0,...,a_m\in\mathcal{A}$ we have $\phi_0(a_0)=res_{r=(1-q)/2}\phi_0^r(a_0)=\tau_{-1}(\gamma a_0)$ and for $m\geq 2$

$$\begin{aligned}\phi_m(a_0,...,a_m) &= res_{r=(1-q)/2}\phi_m^r(a_0,...,a_m)=\sum_{|k|=0}^{2N-m}(-1)^{|k|}\alpha(k)\times\\ &\times\sum_{j=1}^{|k|+m/2}\sigma_{(|k|+m/2),j}\tau_{j-1}\left(\gamma a_0[\mathcal{D},a_1]^{(k_1)}\cdots[\mathcal{D},a_m]^{(k_m)}(1+\mathcal{D}^2)^{-|k|-m/2}\right),\end{aligned}$$

and $(\phi_m)_{m=0,even}^{2N}$ is a (b,B) cocycle for $\mathcal{A}$. When $[q]=2n+1$ is odd, the term with $m=2N$ is zero, and for $m=0,2,...,2N-2$, all the top terms with $|k|=2N-m$ are zero.

Corollary 6.6. *For $1\leq q<2$, the statements in* 3) *of Theorem 6.3 are true without the assumption of isolated dimension spectrum.*

Proposition 6.7 ([CPRS4])**.** *For a QC^∞ finitely summable spectral triple with isolated spectral dimension, the residue cocycle of Theorems 6.3 and 6.5 part 3 represents the class of the Chern character in the (b,B)-bicomplex.*

Computing the cocycle given by the local index formula is often much easier than computing the Fredholm module version. Understanding *all* the terms and interpreting what they tell us about the 'geometry' of a spectral triple is a major undertaking, involving the construction of many new examples.

6.2 The JLO cocycle for θ-summable spectral triples

For θ-summable spectral triples Connes introduced entire cyclic cohomology, see Appendix B, and an appropriate Chern character. A better representative of this Chern character was discovered by Jaffe, Lesniewski and Osterwalder, the JLO cocycle. It is given on even spectral triples by an infinite sequence of cochains $(JLO_{2k})_{k\geq 0}$ defined by

$$JLO_{2k}(a_0, a_1, ..., a_{2k}) = \int_\Delta \mathrm{Trace}(\gamma a_0 e^{-t_0\mathcal{D}^2}[\mathcal{D}, a_1]e^{-t_1\mathcal{D}^2} \cdots e^{-t_{2k-1}\mathcal{D}^2}[\mathcal{D}, a_{2k}]e^{-t_{2k}\mathcal{D}^2})dt_0 dt_1 \cdots dt_{2k}.$$

Here $\Delta = \{(t_0, t_1, \ldots, t_{2k}) \in \mathbb{R}^{2k+1} : t_j \geq 0,\ t_0 + t_1 + \cdots + t_{2k} = 1\}$ is the standard simplex.

In the odd case we have $(JLO_{2k+1})_{k\geq 0}$ defined by

$$JLO_{2k+1}(a_0, a_1, ..., a_{2k+1}) = \sqrt{2\pi i}\int_\Delta \mathrm{Trace}(a_0 e^{-t_0\mathcal{D}^2}[\mathcal{D}, a_1]e^{-t_1\mathcal{D}^2} \cdots e^{-t_{2k}\mathcal{D}^2}[\mathcal{D}, a_{2k}]e^{-t_{2k+1}\mathcal{D}^2})dt_0 dt_1 \cdots dt_{2k+1}.$$

In the context of entire cyclic cohomology, the JLO cocycle represents the Chern character. That is if $[p] \in K_0(\mathcal{A})$ and $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is θ-summable, then

$$\langle [p], [(\mathcal{A}, \mathcal{H}, \mathcal{D})]\rangle = \langle [Ch(p)], [JLO(\mathcal{A}, \mathcal{H}, \mathcal{D})]\rangle = \sum_{k=0}^{\infty} JLO_{2k}(Ch_{2k}(p)).$$

A similar statement holds in the odd case.

The proof in [CP2] is very similar to that used in proving the local index theorem in [CPRS2, CPRS3]. The problem with the JLO cocycle is that it is generically not computable. The local index formula is vastly superior in this regard.

Example 23. Block and Fox showed, [1], starting with the JLO cocycle and using Getzler scaling, that the Chern character for the Dirac operator on a compact spin manifold M can be represented by

$$Ch_k(f_0, f_1, ..., f_k) = c\int_M \hat{A} f_0\, df_1 \wedge \cdots df_k, \quad f_j \in C^\infty(M)$$

yielding the Atiyah-Singer index theorem.

Research exercise Adapt Block annd Fox's proof to start with the local index formula and obtain the Atiyah-Singer formula.

Example 24. The local index cocycle for the noncommutative torus. From known computations of the cyclic cohomology of the noncommutative torus, the cocycle arising from the local index formula must be a linear combination of the 0-cocycle τ_0 and the 2-cocycle τ_2 given by

$$\tau_0(a_0) = \tau(a_0), \qquad \tau_2(a_0, a_1, a_2) = \tau\left(a_0\left(\delta_1(a_1)\delta_2(a_2) - \delta_2(a_1)\delta_1(a_2)\right)\right).$$

Exercise What is the linear combination? *Hint*: The index pairing with any projection is an integer. Consider $1 \in A_\theta$ and the Powers-Rieffel projector p_θ. What integers should you get?

The reason so much effort is made to compute the index pairing in cyclic theory is highlighted by the last exercise. Cyclic cohomology can often be computed using long exact sequences and the relation to Hochschild cohomology. These tools are not as useful as the six term sequences in K-theory/K-homology, and sadly are beyond the scope of these notes.

Chapter 7

Semifinite spectral triples and beyond

7.1 Semifinite spectral triples

We begin with some semifinite versions of standard definitions and results. Let τ be a fixed faithful, normal, semifinite trace on the von Neumann algebra $\mathcal{N}$. Let $\mathcal{K}_{\mathcal{N}}$ be the τ-compact operators in $\mathcal{N}$ (that is the norm closed ideal generated by the projections $E \in \mathcal{N}$ with $\tau(E) < \infty$).

Definition 7.1. *A semifinite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is given by a Hilbert space $\mathcal{H}$, a $*$-algebra $\mathcal{A} \subset \mathcal{N}$ where $\mathcal{N}$ is a semifinite von Neumann algebra acting on $\mathcal{H}$, and a densely defined unbounded self-adjoint operator $\mathcal{D}$ affiliated to $\mathcal{N}$ such that*

1) $[\mathcal{D}, a]$ is densely defined and extends to a bounded operator for all $a \in \mathcal{A}$

2) $a(\lambda - \mathcal{D})^{-1} \in \mathcal{K}_{\mathcal{N}}$ for all $\lambda \notin \mathbb{R}$ and all $a \in \mathcal{A}$.

3) The triple is said to be even if there is $\Gamma \in \mathcal{N}$ such that $\Gamma^* = \Gamma$, $\Gamma^2 = 1$, $a\Gamma = \Gamma a$ for all $a \in \mathcal{A}$ and $\mathcal{D}\Gamma + \Gamma\mathcal{D} = 0$. Otherwise it is odd.

Along with the notion of τ-compact, we naturally get a notion of τ-Fredholm: $T \in \mathcal{N}$ is τ-Fredholm if and only if T is invertible modulo $\mathcal{K}_{\mathcal{N}}$. The index of such operators is in general real valued, but we can often constrain the values...more on this later. Index pairings with K-theory still make sense, and we are still interested in computing such pairings.

Observe that while we can define a semifinite Fredholm module in a similar way, it is not at all clear at this point what the relation to K-homology is, if any.

Definition 7.2. *A semifinite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is QC^k for $k \geq 1$ (Q for quantum) if for all $a \in \mathcal{A}$ the operators a and $[\mathcal{D}, a]$ are in the domain of δ^k, where $\delta(T) = [|\mathcal{D}|, T]$ is the partial derivation on $\mathcal{N}$ defined by $|\mathcal{D}|$. We say that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is QC^∞ if it is QC^k for all $k \geq 1$.*

Note. The notation is meant to be analogous to the classical case, but we introduce the Q so that there is no confusion between quantum differentiability of $a \in \mathcal{A}$ and classical differentiability of functions.

Remarks concerning derivations and commutators. By partial derivation we mean that δ is defined on some subalgebra of $\mathcal{N}$ which need not be (weakly) dense in $\mathcal{N}$. More precisely, dom $\delta = \{T \in \mathcal{N} : \delta(T)$ is bounded$\}$. We also note that if $T \in \mathcal{N}$, one can show that $[|\mathcal{D}|, T]$ is bounded if and only if $[(1+\mathcal{D}^2)^{1/2}, T]$ is bounded, by using the functional calculus to show that $|\mathcal{D}| - (1+\mathcal{D}^2)^{1/2}$ extends to a bounded operator in $\mathcal{N}$. In fact, writing $|\mathcal{D}|_1 = (1+\mathcal{D}^2)^{1/2}$ and $\delta_1(T) = [|\mathcal{D}|_1, T]$ we have

$$\text{dom } \delta^n = \text{dom } \delta_1^n \quad \text{for all } n.$$

We also observe that if $T \in \mathcal{N}$ and $[\mathcal{D}, T]$ is bounded, then $[\mathcal{D}, T] \in \mathcal{N}$. Similar comments apply to $[|\mathcal{D}|, T]$, $[(1+\mathcal{D}^2)^{1/2}, T]$. The proofs can be found in [CPRS2].

7.1.1 Nonunitality

The examples coming from graph algebras, described soon, are often nonunital. Here is a brief summary of what we require in this case. See [R2, R3] and [GGISV] for more information.

Whilst smoothness does not depend on whether $\mathcal{A}$ is unital or not, many analytical problems arise because of the lack of a unit. As in [GGISV, R2, R3], we make two definitions to address these issues.

Definition 7.3. *An algebra $\mathcal{A}$ has local units if for every finite subset of elements $\{a_i\}_{i=1}^n \subset \mathcal{A}$, there exists $\phi \in \mathcal{A}$ such that for each i*

$$\phi a_i = a_i \phi = a_i.$$

Definition 7.4. *Let $\mathcal{A}$ be a Fréchet algebra and $\mathcal{A}_c \subseteq \mathcal{A}$ be a dense subalgebra with local units. Then we call $\mathcal{A}$ a quasi-local algebra (when $\mathcal{A}_c$ is understood.) If $\mathcal{A}_c$ is a dense ideal with local units, we call $\mathcal{A}_c \subset \mathcal{A}$ local.*

Separable quasi-local algebras have an approximate unit $\{\phi_n\}_{n\geq 1} \subset \mathcal{A}_c$ such that for all n, $\phi_{n+1}\phi_n = \phi_n$, [R2]; we call this a local approximate unit.

We also require that when we have a spectral triple the operator $\mathcal{D}$ is compatible with the quasi-local structure of the algebra, in the following sense.

Definition 7.5. *If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a spectral triple, then we define $\Omega^*_{\mathcal{D}}(\mathcal{A})$ to be the algebra generated by $\mathcal{A}$ and $[\mathcal{D}, \mathcal{A}]$.*

Definition 7.6. *A local spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a spectral triple with $\mathcal{A}$ quasi-local such that there exists an approximate unit $\{\phi_n\} \subset \mathcal{A}_c$ for $\mathcal{A}$ satisfying*

$$\Omega^*_{\mathcal{D}}(\mathcal{A}_c) = \bigcup_n \Omega^*_{\mathcal{D}}(\mathcal{A})_n, \quad \text{where}$$

$$\Omega^*_{\mathcal{D}}(\mathcal{A})_n = \{\omega \in \Omega^*_{\mathcal{D}}(\mathcal{A}) : \phi_n \omega = \omega \phi_n = \omega\}.$$

Remark A local spectral triple has a local approximate unit $\{\phi_n\}_{n\geq 1} \subset \mathcal{A}_c$ such that $\phi_{n+1}\phi_n = \phi_n\phi_{n+1} = \phi_n$ and $\phi_{n+1}[\mathcal{D}, \phi_n] = [\mathcal{D}, \phi_n]\phi_{n+1} = [\mathcal{D}, \phi_n]$, see [R2, R3]. We require this property to prove the summability results we require.

7.1.2 Semifinite summability

In the following, let $\mathcal{N}$ be a semifinite von Neumann algebra with faithful normal trace τ. Recall from [FK] that if $S \in \mathcal{N}$, the *t-th generalized singular value* of S for each real $t > 0$ is given by

$$\mu_t(S) = \inf\{||SE|| \ : \ E \text{ is a projection in } \mathcal{N} \text{ with } \tau(1-E) \leq t\}.$$

The ideal $\mathcal{L}^1(\mathcal{N})$ consists of those operators $T \in \mathcal{N}$ such that $\| \ T \ \|_1 := \tau(|T|) < \infty$ where $|T| = \sqrt{T^*T}$. In the Type I setting this is the usual trace class ideal. We will simply write $\mathcal{L}^1$ for this ideal in order to simplify the notation, and denote the norm on $\mathcal{L}^1$ by $\| \cdot \|_1$. An alternative definition in terms of singular values is that $T \in \mathcal{L}^1$ if $\|T\|_1 := \int_0^\infty \mu_t(T)dt < \infty$.

Note that in the case where $\mathcal{N} \neq \mathcal{B}(\mathcal{H})$, $\mathcal{L}^1$ is not complete in this norm but it is complete in the norm $||.||_1 + ||.||_\infty$. (where $||.||_\infty$ is the uniform norm). Another important ideal for us is the domain of the Dixmier trace:

$$\mathcal{L}^{(1,\infty)}(\mathcal{N}) = \left\{ T \in \mathcal{N} \ : \|T\|_{_{\mathcal{L}^{(1,\infty)}}} := \sup_{t>0} \frac{1}{\log(1+t)} \int_0^t \mu_s(T)ds < \infty \right\}.$$

We will suppress the $(\mathcal{N})$ in our notation for these ideals, as $\mathcal{N}$ will always be clear from context. The reader should note that $\mathcal{L}^{(1,\infty)}$ is often taken to mean an ideal in the algebra $\widetilde{\mathcal{N}}$ of τ-measurable operators affiliated to $\mathcal{N}$, [FK]. Our notation is however consistent with that of [C1] in the special case $\mathcal{N} = \mathcal{B}(\mathcal{H})$. With this convention the ideal of τ-compact operators, $\mathcal{K}(\mathcal{N})$, consists of those $T \in \mathcal{N}$ (as opposed to $\widetilde{\mathcal{N}}$) such that

$$\mu_\infty(T) := \lim_{t\to\infty} \mu_t(T) = 0.$$

Definition 7.7. *A semifinite local spectral triple is*

- *finitely summable if there is some* $s_0 \in [0,\infty)$ *such that for all* $s > s_0$ *we have*

$$\tau(a(1+\mathcal{D}^2)^{-s/2}) < \infty \quad \textit{for all } a \in \mathcal{A}_c;$$

- (p,∞)*-summable if*

$$a(\mathcal{D}-\lambda)^{-1} \in \mathcal{L}^{(p,\infty)} \quad \textit{for all } a \in \mathcal{A}_c, \quad \lambda \in \mathbb{C} \setminus \mathbb{R};$$

- θ*-summable if for all* $t > 0$ *we have*

$$\tau(ae^{-t\mathcal{D}^2}) < \infty \quad \textit{for all } a \in \mathcal{A}_c.$$

Remark If $\mathcal{A}$ is unital, and $(1+\mathcal{D}^2)^{-1}$ is τ-compact, $\ker \mathcal{D}$ is τ-finite dimensional. Note that the summability requirements are only for $a \in \mathcal{A}_c$. We do not assume that elements of the algebra $\mathcal{A}$ are all integrable in the nonunital case.

We need to briefly discuss the Dixmier trace, but fortunately we will usually be applying it in reasonably simple situations. For more information on semifinite Dixmier traces, see [CPS2]. For $T \in \mathcal{L}^{(1,\infty)}$, $T \geq 0$, the function

$$F_T : t \to \frac{1}{\log(1+t)} \int_0^t \mu_s(T)ds$$

is bounded. For certain generalised limits, [CPS2], $\omega \in L^\infty(\mathbb{R}_*^+)^*$, we obtain a positive functional on $\mathcal{L}^{(1,\infty)}$ by setting

$$\tau_\omega(T) = \omega(F_T).$$

This is the Dixmier trace associated to the semifinite normal trace τ, denoted τ_ω, and we extend it to all of $\mathcal{L}^{(1,\infty)}$ by linearity, where of course it is a trace. The Dixmier trace τ_ω is defined on the ideal $\mathcal{L}^{(1,\infty)}$, and vanishes on the ideal of trace class operators. Whenever the function F_T has a limit at infinity, all Dixmier traces return the value of the limit. We denote the common value of all Dixmier traces on measurable operators by $⨍$. So if $T \in \mathcal{L}^{(1,\infty)}$ is measurable, for any allowed functional $\omega \in L^\infty(\mathbb{R}_*^+)^*$ we have

$$\tau_\omega(T) = \omega(F_T) = ⨍ T.$$

Example Let $\mathcal{D} = \frac{1}{i}\frac{d}{d\theta}$ act on $L^2(S^1)$. Then it is well known that the spectrum of $\mathcal{D}$ consists of eigenvalues $\{n \in \mathbb{Z}\}$, each with multiplicity one. So, using the standard operator trace, the function $F_{(1+\mathcal{D}^2)^{-1/2}}$ is

$$N \to \frac{1}{\log 2N+1} \sum_{n=-N}^{N} (1+n^2)^{-1/2}$$

which is bounded. So $(1+\mathcal{D}^2)^{-1/2} \in \mathcal{L}^{(1,\infty)}$ and for any Dixmier trace Trace_ω

$$\mathrm{Trace}_\omega((1+\mathcal{D}^2)^{-1/2}) = ⨍ (1+\mathcal{D}^2)^{-1/2} = 2.$$

In [R2, R3] we proved numerous properties of local algebras. The introduction of quasi-local algebras in [GGISV] led us to review the validity of many of these results for quasi-local algebras. Most of the summability results of [R2] are valid in the quasi-local setting. In addition, the summability results of [R3] are also valid for general semifinite spectral triples since they rely only on properties of the ideals $\mathcal{L}^{(p,\infty)}$, $p \geq 1$, [C1, CPS2], and the trace property. We quote the version of the summability results from [R3] that we require below, stated just for $p = 1$. This is a nonunital analogue of a result from [CPS2].

Proposition 7.8 ([R3])**.** *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a QC^∞, local $(1,\infty)$-summable semifinite spectral triple relative to $(\mathcal{N}, \tau)$. Let $T \in \mathcal{N}$ satisfy $T\phi = \phi T = T$ for some $\phi \in \mathcal{A}_c$. Then*

$$T(1+\mathcal{D}^2)^{-1/2} \in \mathcal{L}^{(1,\infty)}.$$

For $Re(s) > 1$, $T(1+\mathcal{D}^2)^{-s/2}$ is trace class. If the limit

$$\lim_{s \to 1/2^+} (s-1/2)\tau(T(1+\mathcal{D}^2)^{-s}) \tag{7.1}$$

exists, then it is equal to

$$\frac{1}{2} ⨍ T(1+\mathcal{D}^2)^{-1/2}.$$

In addition, for any Dixmier trace τ_ω, the function

$$a \mapsto \tau_\omega(a(1+\mathcal{D}^2)^{-1/2})$$

defines a trace on $\mathcal{A}_c \subset \mathcal{A}$.

The various analytic formulae for computing the index, the local index formula, JLO cocycle, Chern characters and the results connecting them all continue to hold in the semifinite case, [CP1, CP2, CPS2, CPRS1, CPRS2, CPRS3, CPRS4]. Just replace the operator trace by a general semifinite trace. In addition, the local index formula holds for local spectral triples, semifinite or not, [R3].

7.2 Graph and k-graph algebras

Graph and k-graph algebras are quite a diverse zoo. One can find algebras in these classes (and their relatives like topological graph algebras, Cuntz-Krieger algebras etc etc) with almost any property you could want. This makes them a great laboratory.

We will not describe k-graph algebras here, or the associated index theory, since it is broadly similar to what follows for the graph(= 1-graph) case. See [PRS] for more details. The following account of semifinite spectral triples for graph algebras comes from [PR].

For a more detailed introduction to graph C^*-algebras we refer the reader to [BPRS, KPR] and the references therein. A directed graph $E = (E^0, E^1, r, s)$ consists of countable sets E^0 of vertices and E^1 of edges, and maps $r, s : E^1 \to E^0$ identifying the range and source of each edge. **We will always assume that the graph is row-finite** which means that each vertex emits at most finitely many edges. Later we will also assume that the graph is *locally finite* which means it is row-finite and each vertex receives at most finitely many edges. We write E^n for the set of paths $\mu = \mu_1\mu_2 \cdots \mu_n$ of length $|\mu| := n$; that is, sequences of edges μ_i such that $r(\mu_i) = s(\mu_{i+1})$ for $1 \leq i < n$. The maps r, s extend to $E^* := \bigcup_{n\geq 0} E^n$ in an obvious way. A *loop* in E is a path $L \in E^*$ with $s(L) = r(L)$, we say that a loop L has an exit if there is $v = s(L_i)$ for some i which emits more than one edge. If $V \subseteq E^0$ then we write $V \geq w$ if there is a path $\mu \in E^*$ with $s(\mu) \in V$ and $r(\mu) = w$ (we also sometimes say that w is downstream from V). A *sink* is a vertex $v \in E^0$ with $s^{-1}(v) = \emptyset$, a *source* is a vertex $w \in E^0$ with $r^{-1}(w) = \emptyset$.

A *Cuntz-Krieger E-family* in a C^*-algebra B consists of mutually orthogonal projections $\{p_v : v \in E^0\}$ and partial isometries $\{S_e : e \in E^1\}$ satisfying the *Cuntz-Krieger relations*

$$S_e^* S_e = p_{r(e)} \text{ for } e \in E^1 \text{ and } p_v = \sum_{\{e:s(e)=v\}} S_e S_e^* \text{ whenever } v \text{ is not a sink.}$$

It is proved in [KPR, Theorem 1.2] that there is a universal C^*-algebra $C^*(E)$ generated by a non-zero Cuntz-Krieger E-family $\{S_e, p_v\}$. A product $S_\mu := S_{\mu_1} S_{\mu_2} \ldots S_{\mu_n}$ is non-zero precisely when $\mu = \mu_1\mu_2 \cdots \mu_n$ is a path in E^n. Since the Cuntz-Krieger relations imply that the projections $S_e S_e^*$ are also mutually orthogonal, we have $S_e^* S_f = 0$ unless $e = f$, and words in $\{S_e, S_f^*\}$ collapse to products of the form $S_\mu S_\nu^*$ for $\mu, \nu \in E^*$ satisfying $r(\mu) = r(\nu)$ (cf. [KPR, Lemma 1.1]). Indeed, because the family $\{S_\mu S_\nu^*\}$ is closed under multiplication and involution, we have

$$C^*(E) = \overline{\text{span}}\{S_\mu S_\nu^* : \mu, \nu \in E^* \text{ and } r(\mu) = r(\nu)\}. \tag{7.2}$$

The algebraic relations and the density of $\text{span}\{S_\mu S_\nu^*\}$ in $C^*(E)$ play a critical role. We adopt the conventions that vertices are paths of length 0, that $S_v := p_v$ for $v \in E^0$, and that all paths μ, ν appearing in (7.2) are non-empty; we recover S_μ, for example, by taking $\nu = r(\mu)$, so that $S_\mu S_\nu^* = S_\mu p_{r(\mu)} = S_\mu$.

If $z \in S^1$, then the family $\{zS_e, p_v\}$ is another Cuntz-Krieger E-family which generates $C^*(E)$, and the universal property gives a homomorphism $\gamma_z : C^*(E) \to C^*(E)$ such that $\gamma_z(S_e) = zS_e$ and $\gamma_z(p_v) = p_v$. The homomorphism $\gamma_{\overline{z}}$ is an inverse for γ_z, so $\gamma_z \in \operatorname{Aut} C^*(E)$, and a routine $\epsilon/3$ argument using (7.2) shows that γ is a strongly continuous action of S^1 on $C^*(E)$. It is called the *gauge action*. Because S^1 is compact, averaging over γ with respect to the normalised Haar measure gives an expectation Φ of $C^*(E)$ onto the fixed-point algebra $C^*(E)^\gamma$:

$$\Phi(a) := \frac{1}{2\pi} \int_{S^1} \gamma_z(a)\, d\theta \quad \text{for} \quad a \in C^*(E), \quad z = e^{i\theta}.$$

The map Φ is positive, has norm 1, and is faithful in the sense that $\Phi(a^*a) = 0$ implies $a = 0$.

From Equation (7.2), it is easy to see that a graph C^*-algebra is unital if and only if the underlying graph is finite. When we consider infinite graphs, we always obtain a quasi-local algebra.

Example For a graph C^*-algebra $A = C^*(E)$, Equation (7.2) shows that

$$A_c = \operatorname{span}\{S_\mu S_\nu^* : \mu, \nu \in E^* \text{ and } r(\mu) = r(\nu)\}$$

is a dense subalgebra. It has local units because

$$p_v S_\mu S_\nu^* = \begin{cases} S_\mu S_\nu^* & v = s(\mu) \\ 0 & \text{otherwise} \end{cases}.$$

Similar comments apply to right multiplication by $p_{s(\nu)}$. By summing the source and range projections (without repetitions) of all $S_{\mu_i} S_{\nu_i}^*$ appearing in a finite sum

$$a = \sum_i c_{\mu_i,\nu_i} S_{\mu_i} S_{\nu_i}^*$$

we obtain a local unit for $a \in A_c$. By repeating this process for any finite collection of such $a \in A_c$ we see that A_c has local units.

7.3 Graph C^*-algebras with semifinite graph traces

This section considers the existence of (unbounded) traces on graph algebras. We denote by A^+ the positive cone in a C^*-algebra A, and we use extended arithmetic on $[0, \infty]$ so that $0 \times \infty = 0$. From [PhR] we take the basic definition:

Definition 7.9. *A trace on a C^*-algebra A is a map $\tau : A^+ \to [0, \infty]$ satisfying*

1) $\tau(a + b) = \tau(a) + \tau(b)$ for all $a, b \in A^+$

2) $\tau(\lambda a) = \lambda\tau(a)$ for all $a \in A^+$ and $\lambda \geq 0$

3) $\tau(a^*a) = \tau(aa^*)$ for all $a \in A$

We say: that τ is faithful if $\tau(a^*a) = 0 \Rightarrow a = 0$; that τ is semifinite if $\{a \in A^+ : \tau(a) < \infty\}$ is norm dense in A^+ (or that τ is densely defined); that τ is lower semicontinuous if whenever $a = \lim_{n\to\infty} a_n$ in norm in A^+ we have $\tau(a) \leq \liminf_{n\to\infty} \tau(a_n)$.

We may extend a (semifinite) trace τ by linearity to a linear functional on (a dense subspace of) A. Observe that the domain of definition of a densely defined trace is a two-sided ideal $I_\tau \subset A$.

Lemma 7.10. *Let E be a row-finite directed graph and let $\tau : C^*(E) \to \mathbb{C}$ be a semifinite trace. Then the dense subalgebra*

$$A_c := \mathrm{span}\{S_\mu S_\nu^* : \mu, \nu \in E^*\}$$

is contained in the domain I_τ of τ.

It is convenient to denote by $A = C^*(E)$ and $A_c = \mathrm{span}\{S_\mu S_\nu^* : \mu, \nu \in E^*\}$.

Lemma 7.11. *Let E be a row-finite directed graph.*

(i) *If $C^*(E)$ has a faithful semifinite trace then no loop can have an exit.*

(ii) *If $C^*(E)$ has a gauge-invariant, semifinite, lower semicontinuous trace τ then $\tau \circ \Phi = \tau$ and*

$$\tau(S_\mu S_\nu^*) = \delta_{\mu,\nu}\tau(p_{r(\mu)}).$$

In particular, τ is supported on $C^*(\{S_\mu S_\mu^* : \mu \in E^*\})$.

Whilst the condition that no loop has an exit is necessary for the existence of a faithful semifinite trace, it is not sufficient.

One of the advantages of graph C^*-algebras is the ability to use both graphical and analytical techniques. There is an analogue of the above discussion of traces in terms of the graph.

Definition 7.12 (cf. [T]). *If E is a row-finite directed graph, then a graph trace on E is a function $g : E^0 \to \mathbb{R}^+$ such that for any $v \in E^0$ we have*

$$g(v) = \sum_{s(e)=v} g(r(e)). \tag{7.3}$$

If $g(v) \neq 0$ for all $v \in E^0$ we say that g is faithful.

Remark One can show by induction that if g is a graph trace on a directed graph with no sinks, and $n \geq 1$

$$g(v) = \sum_{s(\mu)=v,\ |\mu|=n} g(r(\mu)). \tag{7.4}$$

For graphs with sinks, we must also count paths of length at most n which end on sinks. To deal with this more general case we write

$$g(v) = \sum_{s(\mu)=v,\ |\mu|\preceq n} g(r(\mu)) \geq \sum_{s(\mu)=v,\ |\mu|=n} g(r(\mu)), \tag{7.5}$$

where $|\mu| \preceq n$ means that μ is of length n or is of length less than n and terminates on a sink.

As with traces on $C^*(E)$, it is easy to see that a necessary condition for E to have a faithful graph trace is that no loop has an exit.

Proposition 7.13. *Let E be a row-finite directed graph. Then there is a one-to-one correspondence between faithful graph traces on E and faithful, semifinite, lower semicontinuous, gauge invariant traces on $C^*(E)$.*

There are several steps in the construction of a spectral triple. We begin in Subsection 7.3.1 by constructing a C^*-module. We define an unbounded operator $\mathcal{D}$ on this C^*-module as the generator of the gauge action of S^1 on the graph algebra. We show in Subsection 7.3.2 that $\mathcal{D}$ is a regular self-adjoint operator on the C^*-module. We use the phase of $\mathcal{D}$ to construct a Kasparov module.

7.3.1 Building a C^*-module

If you are not familiar with C^*-modules, just think of a Hilbert space, except the inner product takes values in a C^*-algebra, which acts on the right of the module. The examples below are straightforward, and more information can be found in [La, RW].

The important things to know concern operators on these modules which commute with the right action of the C^*-algebra.

Not all F-linear maps $X \to X$ possess adjoints for the inner product. The collection of adjointable endomorphisms (those with an adjoint) is denoted $End_F(X)$. The adjointable endomorphisms form a C^*-algebra with respect to the adjoint operation and operator norm.

Amongst these endomorphisms are the rank one endomorphisms $\Theta_{x,y}$, x, $y \in X$, defined on $z \in X$ by

$$\Theta_{x,y} z := x(y|z)_R.$$

Exercise What is the adjoint of $\Theta_{x,y}$?

Finite sums of rank one endomorphisms are called finite rank. The finite rank endomorphisms generate a closed ideal in $End_F(X)$. This ideal is called the ideal of compact endomorphisms and is denoted $End^0_F(X)$.

Two important things should be noted:

We have a notion of compact, so we have a notion of Fredholm (invertible modulo compacts), and so we have a notion of index. In this case the index is a difference of two F-modules, and this difference defines an element of $K_0(F)$. See [GVF] for a thorough discussion.

This notion of compactness need have nothing whatsoever to do with the compactness of operators on Hilbert space or even the notion of compactness in semifinite von Neumann algebras. Completely different!!!

The actual C^*-modules we will look at in these notes are fairly simple, so you will not have many problems.

The constructions of this subsection work for any locally finite graph. Let $A = C^*(E)$ where E is any locally finite directed graph. Let $F = C^*(E)^\gamma$ be the fixed point subalgebra for the gauge action. Finally, let A_c, F_c be the dense subalgebras of A, F given by the (finite) linear span of the generators.

We make A a right inner product F-module. The right action of F on A is by right multiplication. The inner product is defined by

$$(x|y)_R := \Phi(x^*y) \in F.$$

Here Φ is the canonical expectation. It is simple to check the requirements that $(\cdot|\cdot)_R$ defines an F-valued inner product on A. The requirement $(x|x)_R = 0 \Rightarrow x = 0$ follows from the faithfulness of Φ.

Definition 7.14. *Define X to be the C^*-F-module completion of A for the C^*-module norm*

$$\|x\|_X^2 := \|(x|x)_R\|_A = \|(x|x)_R\|_F = \|\Phi(x^*x)\|_F.$$

Define X_c to be the pre-C^*-F_c-module with linear space A_c and the inner product $(\cdot|\cdot)_R$.

Remark Typically, the action of F does not map X_c to itself, so we may only consider X_c as an F_c module. This is a reflection of the fact that F_c and A_c are quasilocal not local.

The inclusion map $\iota : A \to X$ is continuous since

$$\|a\|_X^2 = \|\Phi(a^*a)\|_F \leq \|a^*a\|_A = \|a\|_A^2.$$

We can also define the gauge action γ on $A \subset X$, and as

$$\begin{aligned}\|\gamma_z(a)\|_X^2 &= \|\Phi((\gamma_z(a))^*(\gamma_z(a)))\|_F = \|\Phi(\gamma_z(a^*)\gamma_z(a))\|_F \\ &= \|\Phi(\gamma_z(a^*a))\|_F = \|\Phi(a^*a)\|_F = \|a\|_X^2,\end{aligned}$$

for each $z \in S^1$, the action of γ_z is isometric on $A \subset X$ and so extends to a unitary U_z on X. This unitary is F linear, adjointable, and we obtain a strongly continuous action of S^1 on X, which we still denote by γ.

For each $k \in \mathbb{Z}$, the projection onto the k-th spectral subspace for the gauge action defines an operator Φ_k on X by

$$\Phi_k(x) = \frac{1}{2\pi}\int_{S^1} z^{-k}\gamma_z(x)d\theta, \quad z = e^{i\theta}, \quad x \in X.$$

Observe that on generators we have $\Phi_k(S_\alpha S_\beta^*) = S_\alpha S_\beta^*$ when $|\alpha| - |\beta| = k$ and is zero when $|\alpha| - |\beta| \neq k$. The range of Φ_k is

$$\text{Range } \Phi_k = \{x \in X : \gamma_z(x) = z^k x \ \text{ for all } z \in S^1\}. \tag{7.6}$$

These ranges give us a natural $\mathbb{Z}$-grading of X.

Remark If E is a finite graph with no loops, then for k sufficiently large there are no paths of length k and so $\Phi_k = 0$. This will obviously simplify many of the convergence issues below.

Lemma 7.15. *The operators Φ_k are adjointable endomorphisms of the F-module X such that $\Phi_k^* = \Phi_k = \Phi_k^2$ and $\Phi_k\Phi_l = \delta_{k,l}\Phi_k$. If $K \subset \mathbb{Z}$ then the sum $\sum_{k\in K}\Phi_k$ converges strictly to a projection in the endomorphism algebra. The sum $\sum_{k\in\mathbb{Z}}\Phi_k$ converges to the identity operator on X.*

Corollary 7.16. *Let $x \in X$. Then with $x_k = \Phi_k x$ the sum $\sum_{k\in\mathbb{Z}} x_k$ converges in X to x.*

7.3.2 The Kasparov module

In this subsection we assume that E is locally finite and furthermore has no sources. That is, every vertex receives at least one edge.

Since we have the gauge action defined on X, we may use the generator of this action to define an unbounded operator $\mathcal{D}$. We will not define or study $\mathcal{D}$ from the generator point of view, rather taking a more bare-hands approach. It is easy to check that $\mathcal{D}$ as defined below is the generator of the S^1 action.

The theory of unbounded operators on C^*-modules that we require is all contained in Lance's book, [La, Chapters 9,10]. We quote the following definitions (adapted to our situation).

Definition 7.17. *Let Y be a right C^*-B-module. A densely defined unbounded operator $\mathcal{D} : \text{dom } \mathcal{D} \subset Y \to Y$ is a B-linear operator defined on a dense B-submodule $\text{dom } \mathcal{D} \subset Y$. The operator $\mathcal{D}$ is closed if the graph*

$$G(\mathcal{D}) = \{(x|\mathcal{D}x)_R : x \in \text{dom } \mathcal{D}\}$$

is a closed submodule of $Y \oplus Y$.

If $\mathcal{D} : \text{dom } \mathcal{D} \subset Y \to Y$ is densely defined and unbounded, define a submodule

$$\text{dom } \mathcal{D}^* := \{y \in Y : \exists z \in Y \text{ such that } \forall x \in \text{dom } \mathcal{D}, (\mathcal{D}x|y)_R = (x|z)_R\}.$$

Then for $y \in \text{dom } \mathcal{D}^*$ define $\mathcal{D}^* y = z$. Given $y \in \text{dom } \mathcal{D}^*$, the element z is unique, so $\mathcal{D}^* : \text{dom}\mathcal{D}^* \to Y$, $\mathcal{D}^* y = z$ is well-defined, and moreover is closed.

Definition 7.18. *Let Y be a right C^*-B-module. A densely defined unbounded operator $\mathcal{D} : \text{dom } \mathcal{D} \subset Y \to Y$ is symmetric if for all $x, y \in \text{dom } \mathcal{D}$*

$$(\mathcal{D}x|y)_R = (x|\mathcal{D}y)_R.$$

A symmetric operator $\mathcal{D}$ is self-adjoint if $\text{dom } \mathcal{D} = \text{dom } \mathcal{D}^*$ (and so $\mathcal{D}$ is necessarily closed). A densely defined unbounded operator $\mathcal{D}$ is regular if $\mathcal{D}$ is closed, $\mathcal{D}^*$ is densely defined, and $(1 + \mathcal{D}^*\mathcal{D})$ has dense range.

The extra requirement of regularity is necessary in the C^*-module context for the continuous functional calculus, and is not automatic, [La, Chapter 9].

With these definitions in hand, we return to our C^*-module X.

Proposition 7.19. *Let X be the right C^*-F-module of Definition 7.14. Define $X_\mathcal{D} \subset X$ to be the linear space*

$$X_\mathcal{D} = \{x = \sum_{k\in\mathbb{Z}} x_k \in X : \|\sum_{k\in\mathbb{Z}} k^2 (x_k|x_k)_R\| < \infty\}.$$

For $x = \sum_{k\in\mathbb{Z}} x_k \in X_\mathcal{D}$ define

$$\mathcal{D}x = \sum_{k\in\mathbb{Z}} k x_k.$$

Then $\mathcal{D} : X_\mathcal{D} \to X$ is a self-adjoint regular operator on X.

Remark Any $S_\alpha S_\beta^* \in A_c$ is in $X_\mathcal{D}$ and

$$\mathcal{D}S_\alpha S_\beta^* = (|\alpha| - |\beta|)S_\alpha S_\beta^*.$$

There is a continuous functional calculus for self-adjoint regular operators, [La, Theorem 10.9], and we use this to obtain spectral projections for $\mathcal{D}$ at the C^*-module level. Let $f_k \in C_c(\mathbb{R})$ be 1 in a small neighbourhood of $k \in \mathbb{Z}$ and zero on $(-\infty, k - 1/2] \cup [k + 1/2, \infty)$. Then it is clear that

$$\Phi_k = f_k(\mathcal{D}).$$

That is the spectral projections of $\mathcal{D}$ are the same as the projections onto the spectral subspaces of the gauge action.

The next Lemma is the first place where we need our graph to be locally finite and have no sources.

Lemma 7.20. *Assume that the directed graph E is locally finite and has no sources. For all $a \in A$ and $k \in \mathbb{Z}$, $a\Phi_k \in End_F^0(X)$, the compact endomorphisms of the right F-module X. If $a \in A_c$ then $a\Phi_k$ is finite rank.*

Remark The proof actually shows that for $k \geq 0$

$$\Phi_k = \sum_{|\rho|=k} \Theta^R_{S_\rho, S_\rho}$$

where the sum converges in the strict topology. A similar formula holds for $k < 0$.

Lemma 7.21. *Let E be a locally finite directed graph with no sources. For all $a \in A$, $a(1+\mathcal{D}^2)^{-1/2}$ is a compact endomorphism of the F-module X.*

Proof. First let $a = p_v$ for $v \in E^0$. Then the sum

$$R_{v,N} := p_v \sum_{k=-N}^{N} \Phi_k (1 + k^2)^{-1/2}$$

is finite rank, by Lemma 7.20. We will show that the sequence $\{R_{v,N}\}_{N \geq 0}$ is convergent with respect to the operator norm $\|\cdot\|_{End}$ of endomorphisms of X. Indeed, assuming that $M > N$,

$$\begin{aligned}\|R_{v,N} - R_{v,M}\|_{End} &= \|p_v \sum_{k=-M}^{-N} \Phi_k(1 + k^2)^{-1/2} + p_v \sum_{k=N}^{M} \Phi_k(1 + k^2)^{-1/2}\|_{End} \\ &\leq 2(1 + N^2)^{-1/2} \to 0, \qquad (7.7)\end{aligned}$$

since the ranges of the $p_v\Phi_k$ are orthogonal for different k. Thus, using the argument from Lemma 7.20, $a(1 + \mathcal{D}^2)^{-1/2} \in End_F^0(X)$. Letting $\{a_i\}$ be a Cauchy sequence from A_c, we have

$$\|a_i(1 + \mathcal{D}^2)^{-1/2} - a_j(1 + \mathcal{D}^2)^{-1/2}\|_{End} \leq \|a_i - a_j\|_{End} = \|a_i - a_j\|_A \to 0,$$

since $\|(1 + \mathcal{D}^2)^{-1/2}\| \leq 1$. Thus the sequence $a_i(1 + \mathcal{D}^2)^{-1/2}$ is Cauchy in norm and we see that $a(1 + \mathcal{D}^2)^{-1/2}$ is compact for all $a \in A$. □

It turns out that the previous lemmas have proved that we have a Kasparov module. This is like a fancy version of a Fredholm module, but now instead of a Hilbert space, we have a C^*-module. Just like Fredholm modules and spectral triples, they come in two flavours, even and odd.

Definition 7.22. *An* **odd Kasparov** A-B**-module** *consists of a countably generated ungraded right* B-C^*-*module* E, *with* $\phi : A \to End_B(E)$ *a* $*$-*homomorphism, together with* $P \in End_B(E)$ *such that* $a(P-P^*)$, $a(P^2-P)$, $[P,a]$ *are all compact endomorphisms. Alternatively, for* $V = 2P-1$, $a(V-V^*)$, $a(V^2-1)$, $[V,a]$ *are all compact endomorphisms for all* $a \in A$. *One can modify* P *to* $\tilde{P}$ *so that* $\tilde{P}$ *is self-adjoint;* $\| \tilde{P} \| \leq 1$; $a(P-\tilde{P})$ *is compact for all* $a \in A$ *and the other conditions for* P *hold with* $\tilde{P}$ *in place of* P *without changing the module* E. *If* P *has a spectral gap about* 0 *(as happens in the cases of interest here) then we may and do assume that* $\tilde{P}$ *is in fact a projection without changing the module,* E.

An **even Kasparov** A-B**-module** *has, in addition to the above data, a grading by a self-adjoint endomorphism* Γ *with* $\Gamma^2 = 1$ *and* $\phi(a)\Gamma = \Gamma\phi(a)$, $V\Gamma + \Gamma V = 0$.

Just as suitable equivalence relations turned Fredholm modules into a cohomology theory for C^*-algebras, so too there are relations which turn Kasparov A-B-modules into a *bivariant* theory, $KK^*(A,B)$. This works so that

$$KK^j(A,\mathbb{C}) = K^j(A),\ K\text{-homology}, \qquad KK^j(\mathbb{C},A) = K_j(A),\ K\text{-theory}.$$

By [K], [Lemma 2, Section 7], the pair (ϕ, P) determines a $KK^1(A,B)$ class, and every class has such a representative. The equivalence relation on pairs (ϕ,P) that give KK^1 classes is generated by unitary equivalence $(\phi,P) \sim (U\phi U^*, UPU^*)$ and homology: $(\phi_1,P_1) \sim (\phi_2,P_2)$ if $P_1\phi_1(a) - P_2\phi_2(a)$ is a compact endomorphism for all $a \in A$, see also [K, Section 7].

Just like Fredholm modules, Kasparov modules have an unbounded version as well.

Definition 7.23. *An* **odd unbounded Kasparov** A-B**-module** *consists of a countably generated ungraded right* B-C^*-*module* E, *with* $\phi : A \to End_B(E)$ *a* $*$-*homomorphism, together with an unbounded self-adjoint regular operator* $\mathcal{D} : \mathrm{dom}\mathcal{D} \subset E \to E$ *such that* $[\mathcal{D},a]$ *is bounded for all* a *in a dense* $*$-*subalgebra of* A *and* $a(1+\mathcal{D}^2)^{-1/2}$ *is a compact endomorphism of* E *for all* $a \in A$. *An* **even unbounded Kasparov** A-B**-module** *has, in addition to the previous data, a* $\mathbb{Z}_2$-*grading with* A *even and* $\mathcal{D}$ *odd, as in Definition 7.22.*

So, now we can state a theorem about graph algebras.

Proposition 7.24. *Assume that the directed graph* E *is locally finite and has no sources. Let* $V = \mathcal{D}(1+\mathcal{D}^2)^{-1/2}$. *Then* (X,V) *defines an odd Kasparov module, and so a class in* $KK^1(A,F)$.

Proof. We will use the approach of [K, Section 4]. We need to show that various operators belong to $End^0_F(X)$. First, $V - V^* = 0$, so $a(V-V^*)$ is compact for all $a \in A$. Also $a(1-V^2) = a(1+\mathcal{D}^2)^{-1}$ which is compact from Lemma 7.21 and the boundedness of $(1+\mathcal{D}^2)^{-1/2}$. Finally, we need to show that $[V,a]$ is compact for all $a \in A$. First we suppose that $a = a_m$ is homogenous for the $\mathbb{T}^1$ action. Then

$$\begin{aligned}[V,a] &= [\mathcal{D},a](1+\mathcal{D}^2)^{-1/2} - \mathcal{D}(1+\mathcal{D}^2)^{-1/2}[(1+\mathcal{D}^2)^{1/2},a](1+\mathcal{D}^2)^{-1/2} \\ &= b_1(1+\mathcal{D}^2)^{-1/2} + Vb_2(1+\mathcal{D}^2)^{-1/2},\end{aligned}$$

where $b_1 = [\mathcal{D}, a] = ma$ and $b_2 = [(1+\mathcal{D}^2)^{1/2}, a]$. Provided that $b_2(1+\mathcal{D}^2)^{-1/2}$ is a compact endomorphism, Lemma 7.21 will show that $[V, a]$ is compact for all homogenous a. So consider the action of $[(1+\mathcal{D}^2)^{1/2}, S_\mu S_\nu^*](1+\mathcal{D}^2)^{-1/2}$ on $x = \sum_{k\in\mathbb{Z}} x_k$. We find

$$\begin{aligned}\sum_{k\in\mathbb{Z}}[(1+\mathcal{D}^2)^{1/2}, S_\mu S_\nu^*](1+\mathcal{D}^2)^{-1/2}x_k &= \sum_{k\in\mathbb{Z}}\left((1+(|\mu|-|\nu|+k)^2)^{1/2}-(1+k^2)^{1/2}\right)(1+k^2)^{-1/2}S_\mu S_\nu^* x_k \\ &= \sum_{k\in\mathbb{Z}} f_{\mu,\nu}(k) S_\mu S_\nu^* \Phi_k x. \qquad (7.8)\end{aligned}$$

The function

$$f_{\mu,\nu}(k) = \left((1+(|\mu|-|\nu|+k)^2)^{1/2}-(1+k^2)^{1/2}\right)(1+k^2)^{-1/2}$$

goes to 0 as $k \to \pm\infty$, and as the $S_\mu S_\nu^* \Phi_k$ are finite rank with orthogonal ranges, the sum in (7.8) converges in the endomorphism norm, and so converges to a compact endomorphism. For $a \in A_c$ we write a as a finite linear combination of generators $S_\mu S_\nu^*$, and apply the above reasoning to each term in the sum to find that $[(1+\mathcal{D}^2)^{1/2}, a](1+\mathcal{D}^2)^{-1/2}$ is a compact endomorphism. Now let $a \in A$ be the norm limit of a Cauchy sequence $\{a_i\}_{i\geq 0} \subset A_c$. Then

$$\|[V, a_i - a_j]\|_{End} \leq 2\|a_i - a_j\|_{End} \to 0,$$

so the sequence $[V, a_i]$ is also Cauchy in norm, and so the limit is compact. □

7.4 The gauge spectral triple of a graph algebra

In this section we will construct a semifinite spectral triple for those graph C^*-algebras which possess a faithful gauge invariant trace, τ. Recall from Proposition 7.13 that such traces arise from faithful graph traces.

We will begin with the right F_c module X_c. In order to deal with the spectral projections of $\mathcal{D}$ we will also assume throughout this section that E is locally finite and has no sources. This ensures, by Lemma 7.20 that for all $a \in A$ the endomorphisms $a\Phi_k$ of X are compact endomorphisms.

As in the proof of Proposition 7.13, we define a $\mathbb{C}$-valued inner product on X_c:

$$\langle x, y\rangle := \tau((x|y)_R) = \tau(\Phi(x^*y)) = \tau(x^*y).$$

This inner product is linear in the second variable. We define the Hilbert space $\mathcal{H} = L^2(X, \tau)$ to be the completion of X_c for $\langle\cdot,\cdot\rangle$. We need a few lemmas in order to obtain the ingredients of our spectral triple.

Lemma 7.25. *The C^*-algebra $A = C^*(E)$ acts on $\mathcal{H}$ by an extension of left multiplication. This defines a faithful nondegenerate $*$-representation of A. Moreover, any endomorphism of X leaving X_c invariant extends uniquely to a bounded linear operator on $\mathcal{H}$.*

Lemma 7.26. *Let $\mathcal{H}, \mathcal{D}$ be as above and let $|\mathcal{D}| = \sqrt{\mathcal{D}^*\mathcal{D}} = \sqrt{\mathcal{D}^2}$ be the absolute value of $\mathcal{D}$. Then for $S_\alpha S_\beta^* \in A_c$, the operator $[|\mathcal{D}|, S_\alpha S_\beta^*]$ is well-defined on X_c, and extends to a bounded operator on $\mathcal{H}$ with*

$$\|[|\mathcal{D}|, S_\alpha S_\beta^*]\|_\infty \leq \Big||\alpha| - |\beta|\Big|.$$

Similarly, $\|[\mathcal{D}, S_\alpha S_\beta^*]\|_\infty = \Big||\alpha| - |\beta|\Big|$.

Corollary 7.27. *The algebra A_c is contained in the smooth domain of the derivation δ where for $T \in \mathcal{B}(\mathcal{H})$, $\delta(T) = [|\mathcal{D}|, T]$. That is*

$$A_c \subseteq \bigcap_{n\geq 0} \text{dom } \delta^n.$$

Definition 7.28. *Define the $*$-algebra $\mathcal{A} \subset A$ to be the completion of A_c in the δ-topology. By Lemma 5.6, $\mathcal{A}$ is Fréchet and stable under the holomorphic functional calculus.*

Lemma 7.29. *If $a \in \mathcal{A}$ then $[\mathcal{D}, a] \in \mathcal{A}$ and the operators $\delta^k(a)$, $\delta^k([\mathcal{D}, a])$ are bounded for all $k \geq 0$. If $\phi \in F \subset \mathcal{A}$ and $a \in \mathcal{A}$ satisfy $\phi a = a = a\phi$, then $\phi[\mathcal{D}, a] = [\mathcal{D}, a] = [\mathcal{D}, a]\phi$. The norm closed algebra generated by $\mathcal{A}$ and $[\mathcal{D}, \mathcal{A}]$ is A. In particular, $\mathcal{A}$ is quasi-local.*

We leave the straightforward proofs of these statements to the reader.

7.4.1 Traces and compactness criteria

We still assume that E is a locally finite graph with no sources and that τ is a faithful semifinite lower semicontinuous gauge invariant trace on $C^*(E)$. We will define a von Neumann algebra $\mathcal{N}$ with a faithful semifinite normal trace $\tilde{\tau}$ so that $\mathcal{A} \subset \mathcal{N} \subset \mathcal{B}(\mathcal{H})$, where $\mathcal{A}$ and $\mathcal{H}$ are as defined in the last subsection. Moreover the operator $\mathcal{D}$ will be affiliated to $\mathcal{N}$. The aim of this subsection will then be to prove the following result.

Theorem 7.30. *Let E be a locally finite graph with no sources, and let τ be a faithful, semifinite, gauge invariant, lower semiconitnuous trace on $C^*(E)$. Then $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a QC^∞, $(1, \infty)$-summable, odd, local, semifinite spectral triple (relative to $(\mathcal{N}, \tilde{\tau})$). For all $a \in \mathcal{A}$, the operator $a(1 + \mathcal{D}^2)^{-1/2}$ is not trace class. If $v \in E^0$ has no sinks downstream*

$$\tilde{\tau}_\omega(p_v(1 + \mathcal{D}^2)^{-1/2}) = 2\tau(p_v),$$

where $\tilde{\tau}_\omega$ is any Dixmier trace associated to $\tilde{\tau}$.

We require the definitions of $\mathcal{N}$ and $\tilde{\tau}$, along with some preliminary results.

Definition 7.31. *Let $End_F^{00}(X_c)$ denote the algebra of finite rank operators on X_c acting on $\mathcal{H}$. Define $\mathcal{N} = (End_F^{00}(X_c))''$, and let $\mathcal{N}_+$ denote the positive cone in $\mathcal{N}$.*

Definition 7.32. *Let $T \in \mathcal{N}$ and $\mu \in E^*$. Let $|v|_k =$ the number of paths of length k with range v, and define for $|\mu| \neq 0$*

$$\omega_\mu(T) = \langle S_\mu, TS_\mu\rangle + \frac{1}{|r(\mu)|_{|\mu|}}\langle S_\mu^*, TS_\mu^*\rangle.$$

For $|\mu| = 0$, $S_\mu = p_v$, for some $v \in E^0$, set $\omega_\mu(T) = \langle S_\mu, TS_\mu\rangle$. Define

$$\tilde{\tau} : \mathcal{N}_+ \to [0, \infty], \quad by \quad \tilde{\tau}(T) = \lim_{L\nearrow} \sum_{\mu\in L\subset E^*} \omega_\mu(T)$$

where L is in the net of finite subsets of E^*.

Remark For $T, S \in \mathcal{N}_+$ and $\lambda \geq 0$ we have

$$\tilde{\tau}(T + S) = \tilde{\tau}(T) + \tilde{\tau}(S) \quad \text{and} \quad \tilde{\tau}(\lambda T) = \lambda\tilde{\tau}(T) \text{ where } 0 \times \infty = 0.$$

Proposition 7.33. *The function* $\tilde{\tau}: \mathcal{N}_+ \to [0, \infty]$ *defines a faithful normal semifinite trace on* $\mathcal{N}$. *Moreover,*

$$End_F^{00}(X_c) \subset \mathcal{N}_{\tilde{\tau}} := \text{span}\{T \in \mathcal{N}_+ : \tilde{\tau}(T) < \infty\},$$

the domain of definition of $\tilde{\tau}$, *and*

$$\tilde{\tau}(\Theta^R_{x,y}) = \langle y, x\rangle = \tau(y^* x), \quad x, y \in X_c.$$

Notation If $g : E^0 \to \mathbb{R}_+$ is a faithful graph trace, we shall write τ_g for the associated semifinite trace on $C^*(E)$, and $\tilde{\tau}_g$ for the associated faithful, semifinite, normal trace on $\mathcal{N}$ constructed above.

Lemma 7.34. *Let* E *be a locally finite graph with no sources and a faithful graph trace* g. *Let* $v \in E^0$ *and* $k \in \mathbb{Z}$. *Then*

$$\tilde{\tau}_g(p_v \Phi_k) \leq \tau_g(p_v)$$

with equality when $k \leq 0$ *or when* $k > 0$ *and there are no sinks within* k *vertices of* v.

Proposition 7.35. *Assume that the directed graph* E *is locally finite, has no sources and has a faithful graph trace* g. *For all* $a \in A_c$ *the operator* $a(1 + \mathcal{D}^2)^{-1/2}$ *is in the ideal* $\mathcal{L}^{(1,\infty)}(\mathcal{N}, \tilde{\tau}_g)$.

Remark Using Proposition 7.8, one can check that

$$res_{s=0}\tilde{\tau}_g(p_v(1 + \mathcal{D}^2)^{-1/2-s}) = \frac{1}{2}\tilde{\tau}_{g\omega}(p_v(1 + \mathcal{D}^2)^{-1/2}). \tag{7.9}$$

We will require this formula when we apply the local index theorem.

Corollary 7.36. *Assume* E *is locally finite, has no sources and has a faithful graph trace* g. *Then for all* $a \in A$, $a(1 + \mathcal{D}^2)^{-1/2} \in \mathcal{K}_{\mathcal{N}}$.

7.5 The index pairing

Having constructed semifinite spectral triples for graph C^*-algebras arising from locally finite graphs with no sources and a faithful graph trace, we can apply the semifinite local index theorem described in [CPRS2]. See also [CPRS3, CM, H].

There is a C^*-module index, which takes its values in the K-theory of the core. The numerical index is obtained by applying the trace $\tilde{\tau}$ to the difference of projections representing the K-theory class coming from the C^*-module index.

Thus for any unitary u in a matrix algebra over the graph algebra A

$$\langle [u], [(\mathcal{A}, \mathcal{H}, \mathcal{D})]\rangle \in \tilde{\tau}_*(K_0(F)).$$

We compute this pairing for unitaries arising from loops (with no exit), which provide a set of generators of $K_1(\mathcal{A})$. To describe the K-theory of the graphs we are considering, we employ the notion of ends.

Definition 7.37. *Let E be a row-finite directed graph. An* end *will mean a sink, a loop without exit or an infinite path with no exits.*

Remark We shall identify an end with the vertices which comprise it. Once on an end (of any sort) the graph trace remains constant.

Lemma 7.38. *Let $C^*(E)$ be a graph C^*-algebra such that no loop in the locally finite graph E has an exit. Then,*

$$K_0(C^*(E)) = \mathbb{Z}^{\#ends}, \qquad K_1(C^*(E)) = \mathbb{Z}^{\#loops}.$$

If $A = C^*(E)$ is nonunital, we will denote by A^+ the algebra obtained by adjoining a unit to A; otherwise we let A^+ denote A.

Definition 7.39. *Let E be a locally finite graph such that $C^*(E)$ has a faithful graph trace g. Let L be a loop in E, and denote by $p_1, \dots, p_n$ the projections associated to the vertices of L and $S_1, \dots, S_n$ the partial isometries associated to the edges of L, labelled so that $S_n^*S_n = p_1$ and*

$$S_i^*S_i = p_{i+1},\ i = 1, \dots, n-1, \quad S_iS_i^* = p_i,\ i = 1, \dots, n.$$

Lemma 7.40. *Let $A = C^*(E)$ be a graph C^*-algebra with faithful graph trace g. For each loop L in E we obtain a unitary in A^+,*

$$u = 1 + S_1 + S_2 + \cdots + S_n - (p_1 + p_2 + \cdots + p_n),$$

whose $K_1(A)$ class does not vanish. Moreover, distinct loops give rise to distinct $K_1(A)$ classes, and we obtain a complete set of generators of $K_1(A)$ in this way.

Proposition 7.41. *Let E be a locally finite graph with no sources and a faithful graph trace g and $A = C^*(E)$. The pairing between the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ of Theorem 7.30 with $K_1(A)$ is given on the generators of Lemma 7.40 by*

$$\langle [u], [(\mathcal{A}, \mathcal{H}, \mathcal{D})] \rangle = -\sum_{i=1}^{n} \tau_g(p_i) = -n\tau_g(p_1).$$

Proof. The semifinite local index formula, [CPRS2] provides a general formula for the Chern character of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. In our setting it is given by a one-cochain

$$\phi_1(a_0, a_1) = res_{s=0}\sqrt{2\pi i}\tilde{\tau}_g(a_0[\mathcal{D}, a_1](1 + \mathcal{D}^2)^{-1/2-s}),$$

and the pairing (spectral flow) is given by

$$sf(\mathcal{D}, u\mathcal{D}u^*) = \langle [u], (\mathcal{A}, \mathcal{H}, \mathcal{D}) \rangle = \frac{1}{\sqrt{2\pi i}}\phi_1(u, u^*).$$

Now $[\mathcal{D}, u^*] = -\sum S_i^*$ and $u[\mathcal{D}, u^*] = -\sum_{i=1}^n p_i$. Using Equation (7.9) and Proposition 7.35,

$$sf(\mathcal{D}, u\mathcal{D}u^*) = -res_{s=0}\tilde{\tau}_g(\sum_{i=1}^{n} p_i(1 + \mathcal{D}^2)^{-1/2-s}) = -\sum_{i=1}^{n} \tau_g(p_i) = -n\tau_g(p_1),$$

the last equalities following since all the p_i have equal trace and there are no sinks 'downstream' from any p_i, since no loop has an exit. □

Remark The C^*-algebra of the graph consisting of a single edge and single vertex is $C(S^1)$ (we choose Lebesgue measure as our trace, normalised so that $\tau(1) = 1$). For this example, the spectral triple we have constructed is the Dirac triple of the circle, $(C^\infty(S^1), L^2(S^1), \frac{1}{i}\frac{d}{d\theta})$, (as can be seen from Corollary 7.43 below.) The index theorem above gives the correct normalisation for the index pairing on the circle. That is, if we denote by z the unitary coming from the construction of Lemma 7.40 applied to this graph, then $\langle[\bar{z}], (\mathcal{A}, \mathcal{H}, \mathcal{D})\rangle = 1$.

Proposition 7.42. *Let E be a locally finite graph with no sources and a faithful graph trace g, and $A = C^*(E)$. The pairing between the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ of Theorem 7.30 with $K_1(A)$ can be computed as follows. Let P be the positive spectral projection for $\mathcal{D}$, and perform the C^* index pairing [KNR]*

$$K_1(A) \times KK^1(A, F) \to K_0(F), \qquad [u] \times [(X, P)] \to [\ker PuP] - [\mathrm{coker} PuP].$$

Then we have

$$sf(\mathcal{D}, u\mathcal{D}u^*) = \tilde{\tau}_g(\ker PuP) - \tilde{\tau}_g(\mathrm{coker} PuP) = \tilde{\tau}_{g*}([\ker PuP] - [\mathrm{coker} PuP]).$$

Proof. It suffices to prove this on the generators of $K_1(A)$ arising from loops L in E. Let $u = 1 + \sum_i S_i - \sum_i p_i$ be the corresponding unitary in A^+ defined in Lemma 7.40. We will show that $\ker PuP = \{0\}$ and that $\mathrm{coker} PuP = \sum_{i=1}^n p_i \Phi_1$. For $a \in PX$ write $a = \sum_{m \geq 1} a_m$. For each $m \geq 1$ write $a_m = \sum_{i=1}^n p_i a_m + (1 - \sum_{i=1}^n p_i) a_m$. Then

$$\begin{aligned} PuPa_m &= P(1 - \sum_{i=1}^n p_i + \sum_{i=1}^n S_i) a_m \\ &= P(1 - \sum^n p_i + \sum^n S_i)(\sum^n p_i a_m) + P(1 - \sum^n p_i + \sum^n S_i)(1 - \sum^n p_i) a_m \\ &= P \sum^n S_i a_m + P(1 - \sum^n p_i) a_m \\ &= \sum^n S_i a_m + (1 - \sum^n p_i) a_m. \end{aligned}$$

It is clear from this computation that $PuPa_m \neq 0$ for $a_m \neq 0$.

Now suppose $m \geq 2$. If $\sum_{i=1}^n p_i a_m = a_m$ then $a_m = \lim_N \sum_{k=1}^N S_{\mu_k} S_{\nu_k}^*$ with $|\mu_k| - |\nu_k| = m \geq 2$ and $S_{\mu_{k_1}} = S_i$ for some i. So we can construct b_{m-1} from a_m by removing the initial S_i's. Then $a_m = \sum_{i=1}^n S_i b_{m-1}$, and $\sum_{i=1}^n p_i b_{m-1} = b_{m-1}$. For arbitrary a_m, $m \geq 2$, we can write $a_m = \sum_i p_i a_m + (1 - \sum_i p_i) a_m$, and so

$$\begin{aligned} a_m &= \sum^n p_i a_m + (1 - \sum^n p_i) a_m \\ &= \sum^n S_i b_{m-1} + (1 - \sum^n p_i) a_m \quad \text{and by adding zero} \\ &= \sum^n S_i b_{m-1} + (1 - \sum^n p_i) b_{m-1} + \big(\sum^n S_i + (1 - \sum^n p_i)\big)(1 - \sum^n p_i) a_m \\ &= u b_{m-1} + u(1 - \sum^n p_i) a_m \\ &= PuPb_{m-1} + PuP(1 - \sum^n p_i) a_m. \end{aligned}$$

Thus PuP maps onto $\sum_{m \geq 2} \Phi_m X$.

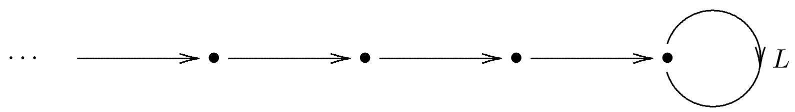
For $m = 1$, if we try to construct b_0 from $\sum_{i=1}^n p_i a_1$ as above, we find $PuPb_0 = 0$ since $Pb_0 = 0$. Thus $\text{coker} PuP = \sum^n p_i \Phi_1 X$. By Proposition 7.41, the pairing is then

$$sf(\mathcal{D}, u\mathcal{D}u^*) = -\sum^n \tau_g(p_i) = -\tilde{\tau}_g(\sum^n p_i\Phi_1)$$
$$= -\tilde{\tau}_{g*}([\text{coker} PuP]) = -\tilde{\tau}_g(\text{coker} PuP). \quad (7.10)$$

Thus we can recover the numerical index using $\tilde{\tau}_g$ and the C^*-index. □

The following example shows that the semifinite index provides finer invariants of directed graphs than those obtained from the ordinary index. The ordinary index computes the pairing between the K-theory and K-homology of $C^*(E)$, while the semifinite index also depends on the core and the gauge action.

Corollary 7.43 (Example)**.** *Let $C^*(E_n)$ be the algebra determined by the graph*



where the loop L has n edges. Then $C^*(E_n) \cong C(S^1) \otimes \mathcal{K}$ for all n, but n is an invariant of the pair of algebras $(C^*(E_n), F_n)$ where F_n is the core of $C^*(E_n)$.

Proof. Observe that the graph E_n has a one parameter family of faithful graph traces, specified by $g(v) = r \in \mathbb{R}_+$ for all $v \in E^0$.

First consider the case where the graph consists only of the loop L. The C^*-algebra A of this graph is isomorphic to $M_n(C(S^1))$, via

$$S_i \to e_{i,i+1},\ i = 1, \ldots, n-1, \quad S_n \to id_{S^1} e_{n,1},$$

where the $e_{i,j}$ are the standard matrix units for $M_n(\mathbb{C})$. The unitary

$$S_1S_2\cdots S_n + S_2S_3\cdots S_1 + \cdots + S_nS_1\cdots S_{n-1}$$

is mapped to the orthogonal sum $id_{S^1}e_{1,1} \oplus id_{S^1}e_{2,2} \oplus \cdots \oplus id_{S^1}e_{n,n}$. The core F of A is $\mathbb{C}^n = \mathbb{C}[p_1, \ldots, p_n]$. Since $KK^1(A, F)$ is equal to

$$\oplus^n KK^1(A, \mathbb{C}) = \oplus^n KK^1(M_n(C(S^1)), \mathbb{C}) = \oplus^n K^1(C(S^1)) = \mathbb{Z}^n$$

we see that n is the rank of $KK^1(A, F)$ and so an invariant, but let us link this to the index computed in Propositions 7.41 and 7.42 more explicitly. Let $\phi : C(S^1) \to A$ be given by $\phi(id_{S^1}) = S_1S_2\cdots S_n \oplus \sum_{i=2}^n e_{i,i}$. We observe that $\mathcal{D} = \sum_{i=1}^n p_i\mathcal{D}$ because the 'off-diagonal' terms are $p_i\mathcal{D}p_j = \mathcal{D}p_ip_j = 0$. Since $S_1S_1^* = S_n^*S_n = p_1$, we find (with P the positive spectral projection of $\mathcal{D}$)

$$\phi^*(X, P) = (p_1X, p_1Pp_1) \oplus \text{degenerate module} \in KK^1(C(S^1), F).$$

Now let $\psi : F \to \mathbb{C}^n$ be given by $\psi(\sum_j z_jp_j) = (z_1, z_2, ..., z_n)$. Then

$$\psi_*\phi^*(X, P) = \oplus_{j=1}^n (p_1Xp_j, p_1Pp_1) \in \oplus^n K^1(C(S^1)).$$

Now $X \cong M_n(C(S^1))$, so $p_1 X p_j \cong C(S^1)$ for each $j = 1, \dots, n$. It is easy to check that $p_1 \mathcal{D} p_1$ acts by $\frac{1}{i}\frac{d}{d\theta}$ on each $p_1 X p_j$, and so our Kasparov module maps to

$$\psi_* \phi^*(X, P) = \oplus^n (C(S^1), P_{\frac{1}{i}\frac{d}{d\theta}}) \in \oplus^n K^1(C(S^1)),$$

where $P_{\frac{1}{i}\frac{d}{d\theta}}$ is the positive spectral projection of $\frac{1}{i}\frac{d}{d\theta}$. The pairing with id_{S^1} is nontrivial on each summand, since $\phi(id_{S^1}) = S_1 \cdots S_n \oplus \sum_{i=2}^n e_{i,i}$ is a unitary mapping $p_1 X p_j$ to itself for each j. So we have, [HR],

$$\begin{aligned} id_{S^1} \times \psi_* \phi^*(X, P) &= \sum_{j=1}^n Index(P id_{S^1} P : p_1 P X p_j \to p_1 P X p_j) \\ &= -\sum_{j=1}^n [p_j] \in K_0(\mathbb{C}^n). \end{aligned} \tag{7.11}$$

By Proposition 7.42, applying the trace to this index gives $-n\tau_g(p_1)$. Of course in Proposition 7.42 we used the unitary $S_1 + S_2 + \cdots + S_n$, however in $K_1(A)$

$$[S_1 S_2 \cdots S_n] = [S_1 + S_2 + \cdots + S_n] = [id_{S^1}].$$

To see this, observe that

$$(S_1 + \cdots + S_n)^n = S_1 S_2 \cdots S_n + S_2 S_3 \cdots S_1 + \cdots + S_n S_1 \cdots S_{n-1}.$$

This is the orthogonal sum of n copies of id_{S^1}, which is equivalent in K_1 to $n[id_{S^1}]$. Finally, $[S_1 + \cdots + S_n] = [id_{S^1}]$ and so

$$[(S_1 + \cdots + S_n)^n] = n[S_1 + \cdots + S_n] = n[id_{S^1}].$$

Since we have cancellation in K_1, this implies that the class of $S_1 + \cdots + S_n$ coincides with the class of $S_1 S_2 \cdots S_n$.

Having seen what is involved, we now add the infinite path on the left. The core becomes $\mathcal{K} \oplus \mathcal{K} \oplus \cdots \oplus \mathcal{K}$ (n copies). Since $A = C(S^1) \otimes \mathcal{K} = M_n(C(S^1)) \otimes \mathcal{K}$, the intrepid reader can go through the details of an argument like the one above, with entirely analogous results. □

Since the invariants obtained from the semifinite index are finer than the isomorphism class of $C^*(E)$, depending as they do on $C^*(E)$ and the gauge action, they can be regarded as invariants of the differential structure. That is, the core F can be recovered from the gauge action, and we regard these invariants as arising from the differential structure defined by $\mathcal{D}$. Thus in this case, the semifinite index produces invariants of the differential topology of the noncommutative space $C^*(E)$.

7.6 The relationship between semifinite triples and KK-theory

In order to construct a semifinite spectral triple for a graph algebra with gauge invariant trace, we first constructed a Kasparov module. The numerical index we computed was then compatible with the Kasparov product (K-theory-valued index). The question is whether this is always the case. The following proposition from [KNR] gives an affirmative answer.

Proposition 7.44. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a unital semifinite spectral triple relative to $(\mathcal{N}, \tau)$. Suppose that the norm closure $A = \overline{\mathcal{A}}$ of $\mathcal{A}$, is a separable C^*-algebra. Let $u \in \mathcal{A}$ be unitary. Set $\mathcal{D}_t = (1-t)\mathcal{D} + tu^*\mathcal{D}u = \mathcal{D} + tu^*[\mathcal{D}, u]$, then the unbounded semifinite spectral flow of the path $t \mapsto \mathcal{D}_t$ is given by*

$$\mathrm{sf}\{\mathcal{D}_t\} = \tau_*\big(Q_{\ker(pup+1-p)}\big) - \tau_*\big(Q_{\ker(pu^*p+1-p)}\big)$$

where $\tau_* : K_0(\mathcal{K}_\mathcal{N}) \to \mathbb{R}$ is the homomorphism induced by the trace τ and $p = \chi_{[0,\infty)}(F_\mathcal{D})$. In addition there exist a separable C^*-algebra $B \subseteq \mathcal{K}_\mathcal{N}$ and a class $[\mathcal{D}_B] \in KK^1(A, B)$ such that

$$\mathrm{sf}\{\mathcal{D}_t\} = \tau\big(i_*([u] \otimes_A [\mathcal{D}_B])\big)$$

where $i : B \to \mathcal{K}_N$ is the inclusion and $[u] \in K_1(A)$ is the class of the unitary.

Thus semifinite index theory turns out to be a special, computable, case of Kasparov theory.

The greater the constraint we can place on the 'right-hand' algebra B, the more constraint we place on the possible values of the index. Since the index is *a priori* any real number, this can be very important.

For the graph of the previous section, the index actually tells us the value of the graph trace on a projection (analytic input), and the number of vertices on the loop (topological data).

7.7 Modular spectral triples, type III von Neumann algebras and KMS states

The Cuntz algebra O_n is a graph algebra and we can construct a Kasparov module. However, it has no traces, so we can not construct a semifinite spectral triple.

The subject of ongoing research at the moment is understanding how to do index theory for KMS states. This has been done for the Cuntz algebra, [CPR2], quantum $SU(2)$, [CRT], and for the general situation where the time evolution is periodic, [CNNR]. More examples where the time evolution is not periodic are being studied.

While this would take us too far afield, the subject is exciting. Connections to quantum statistical mechanics, equivariant KK-theory and numerous other fields are developing. Examples from arithmetic geometry have produced invariants of Mumford curves.

The possibility of obtaining invariants in such a radically diverse collection of examples is, I hope, a sufficiently exciting place to end.

Appendix A

Unbounded operators on Hilbert space

This appendix is stolen primarily from [HR], but also see [RS].

Definition A.1. *An unbounded operator D on a Hilbert space $\mathcal{H}$ is a linear map from a subspace* $\mathrm{Dom}D \subset \mathcal{H}$ *(called the domain of D) to $\mathcal{H}$. The unbounded operator D is said to be densely defined if* $\mathrm{Dom}D$ *is dense in $\mathcal{H}$.*

Remark We are really only interested in densely defined operators.

Definition A.2. *If D, D' are unbounded operators on $\mathcal{H}$ and* $\mathrm{Dom}D \subset \mathrm{Dom}D'$ *and $D\xi = D'\xi$ for all $\xi \in \mathrm{Dom}D$, then we write $D \subseteq D'$ and say that D' is an extension of D.*

Definition A.3. *If D is an (unbounded) operator on $\mathcal{H}$, the graph of D is the subspace $\{(\xi, D\xi) : \xi \in \mathrm{Dom}D\} \subset \mathcal{H} \times \mathcal{H}$. The operator D is said to be closed if the graph is a closed subspace of $\mathcal{H} \times \mathcal{H}$. The operator D is said to be closable if D has a closed extension D'.*

If dom$D$ is all of $\mathcal{H}$ and D is closed, then the closed graph theorem shows that D is bounded. For an unbounded operator D to be closed we must have: whenever $\{\xi_k\}_{k\geq 1} \subset \mathrm{Dom}D$ is a convergent sequence such that $\{D\xi_k\}_{k\geq 1}$ is also a convergent sequence we have $\lim_{k\to\infty} D\xi_k = D \lim_{k\to\infty} \xi_k$.

Any closable operator has a closure $\bar{D} \supseteq D$ which is the operator whose graph is the closure of the graph of D.

Definition A.4. *Let D be an unbounded densely defined operator on $\mathcal{H}$. Define*

$$\mathrm{Dom}D^* = \{\eta \in \mathcal{H} : \forall \xi \in \mathrm{Dom}D\ \exists \rho \in \mathcal{H} \text{ such that } \langle D\xi, \eta\rangle = \langle \xi, \rho\rangle\}.$$

Then we define $D^* : \mathrm{Dom}D^* \to \mathcal{H}$ by $D^*\eta = \rho$. This is well-defined, and the operator D^* is closed.

Exercise Prove the two assertions of the definition.

Definition A.5. *An operator D is symmetric if $D \subseteq D^*$, so*

$$\langle D\xi, \eta\rangle = \langle \xi, D\eta\rangle \qquad \text{for all } \xi, \eta \in \mathrm{Dom}D.$$

The operator D is self-adjoint if $D = D^*$, so D is symmetric and $\mathrm{Dom}D = \mathrm{Dom}D^*$.

Despite appearances, there is a world of difference between symmetric and self-adjoint operators. If D is symmetric then it is closable and $D \subseteq \bar{D} \subseteq D^*$. If Dom$\bar{D}$ = DomD^* then we say that D is essentially self-adjoint, meaning it has a unique self-adjoint extension.

Let D be a closed operator, and give Dom$D$ the graph norm

$$\|\xi\|_D^2 = \|\xi\|^2 + \|D\xi\|^2.$$

Then Dom$D$ is closed in the topology coming from the graph norm. The resolvent set of D is the set of all $\lambda \in \mathbb{C}$ such that the operator

$$(D - \lambda Id_{\mathcal{H}}) : \mathrm{Dom}D \to \mathcal{H}$$

has a two-sided inverse. Any such inverse is a bounded operator from $\mathcal{H}$ to Dom$D$ and so is a bounded operator.

The spectrum of D is the complement of the resolvent set, i.e. those $\lambda \in \mathbb{C}$ such that $(D - \lambda Id_{\mathcal{H}})$ is not invertible.

Lemma A.6. *The spectrum of a self-adjoint operator is real.*

This allows us, after some effort, to come up with a functional calculus for self-adjoint operators. This functional calculus allows us to define $f(D)$ for any bounded Borel function on the spectrum of D. If $f_n \to f$ pointwise, then $f_n(D) \to f(D)$ in the strong operator topology. With suitable care with domains, it is also possible to define unbounded Borel functions of D. For a thorough discussion of this, see [RS].

Two important results for us:

Any differential operator on a manifold-without-boundary is closable.

Every symmetric differential operator on a compact manifold-without-boundary is essentially self-adjoint.

These two results can be found in [HR].

Finally, an unbounded operator $\mathcal{D}$ on a Hilbert space $\mathcal{H}$ is said to be affiliated to a von Neumann algebra $\mathcal{N} \subset \mathcal{B}(\mathcal{H})$ if for all projections $p \in \mathcal{N}'$ we have $p : \mathrm{dom}\mathcal{D} \to \mathrm{dom}\mathcal{D}$ and $\mathcal{D}p = p\mathcal{D}$.

Appendix B

Cyclic cohomology and the Chern character

A central feature of [C1] is the translation of the K-theory pairing to cyclic theory in order to obtain index theorems. One associates to a suitable representative of a K-theory class, respectively a K-homology class, a class in periodic cyclic homology, respectively a class in periodic cyclic cohomology, called a Chern character in both cases. The principal result is then

$$sf(\mathcal{D}, u\mathcal{D}u^*) = \langle [u], [(\mathcal{A}, \mathcal{H}, \mathcal{D})] \rangle = -\frac{1}{\sqrt{2\pi i}} \langle [Ch_*(u)], [Ch^*(\mathcal{A}, \mathcal{H}, \mathcal{D})] \rangle, \tag{B.1}$$

where $[u] \in K_1(\mathcal{A})$ is a K-theory class with representative u and $[(\mathcal{A}, \mathcal{H}, \mathcal{D})]$ is the K-homology class of the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. A similar statement holds in the even case.

On the right hand side, $Ch_*(u)$ is the Chern character of u, and $[Ch_*(u)]$ its periodic cyclic homology class. Similarly $[Ch^*(\mathcal{A}, \mathcal{H}, \mathcal{D})]$ is the periodic cyclic cohomology class of the Chern character of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. The analogue of Equation (B.1), for a suitable cocycle associated to $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, in the general semifinite case is part of the statement of the (semifinite) local index formula.

We will use the normalised (b, B)-bicomplex (see [C1, L]). The reason for this is that one can easily realise the Chern character of a finitely summable Fredholm module, a cyclic cocycle, in the b, B picture, but going the other way requires substantial work, [CPRS4].

We introduce the following linear spaces. Let $C_m = \mathcal{A} \otimes \bar{\mathcal{A}}^{\otimes m}$ where $\bar{\mathcal{A}}$ is the quotient $\mathcal{A}/\mathbb{C}I$ with I being the identity element of $\mathcal{A}$ and (assuming with no loss of generality that $\mathcal{A}$ is complete in the δ-topology) we employ the projective tensor product. Let $C^m = Hom(C_m, \mathbb{C})$ be the linear space of continuous multilinear functionals on C_m. We may define the (b, B) bicomplex using these spaces (as opposed to $C_m = \mathcal{A}^{\otimes m+1}$ et cetera) and the resulting cohomology will be the same. This follows because the bicomplex defined using $\mathcal{A} \otimes \bar{\mathcal{A}}^{\otimes m}$ is quasi-isomorphic to that defined using $\mathcal{A} \otimes \mathcal{A}^{\otimes m}$.

A normalised $(\mathbf{b},\mathbf{B})$**-cochain**, ϕ, is a finite collection of continuous multilinear functionals on $\mathcal{A}$,

$$\phi = \{\phi_m\}_{m=1,2,\dots,M} \text{ with } \phi_m \in C^m.$$

It is a (normalised) $(\mathbf{b},\mathbf{B})$**-cocycle** if, for all m, $b\phi_m + B\phi_{m+2} = 0$ where $b : C^m \to C^{m+1}$, $B : C^m \to C^{m-1}$ are the coboundary operators given by

$$(B\phi_m)(a_0, a_1, \dots, a_{m-1}) = \sum_{j=0}^{m-1} (-1)^{(m-1)j} \phi_m(1, a_j, a_{j+1}, \dots, a_{m-1}, a_0, \dots, a_{j-1})$$

$$(b\phi_{m-2})(a_0, a_1, \dots, a_{m-1}) =$$
$$\sum_{j=0}^{m-2} (-1)^j \phi_{m-2}(a_0, a_1, \dots, a_j a_{j+1}, \dots, a_{m-1}) + (-1)^{m-1}\phi_{m-2}(a_{m-1}a_0, a_1, \dots, a_{m-2})$$

We write $(b + B)\phi = 0$ for brevity. Thought of as functionals on $\mathcal{A}^{\otimes m+1}$ a normalised cocycle will satisfy $\phi(a_0, a_1, \dots, a_n) = 0$ whenever any $a_j = 1$ for $j \geq 1$. An **odd** (**even**) cochain has $\{\phi_m\} = 0$ for m even (odd).

Similarly, a $(\mathbf{b^T},\mathbf{B^T})$**-chain**, c is a (possibly infinite) collection $c = \{c_m\}_{m=1,2,\dots}$ with $c_m \in C_m$. The (b, B)-chain $\{c_m\}$ is a $(\mathbf{b^T},\mathbf{B^T})$**-cycle** if $b^T c_{m+2} + B^T c_m = 0$ for all m. More briefly, we write $(b^T + B^T)c = 0$. Here b^T, B^T are the boundary operators of cyclic homology, and are the transpose of the coboundary operators b, B in the following sense.

The pairing between a (b, B)-cochain $\phi = \{\phi_m\}_{m=1}^M$ and a (b^T, B^T)-chain $c = \{c_m\}$ is given by ($M \in \mathbb{N}$ or $M = \infty$)

$$\langle \phi, c\rangle = \sum_{m=1}^{M} \phi_m(c_m).$$

This pairing satisfies

$$\langle (b + B)\phi, c\rangle = \langle \phi, (b^T + B^T)c\rangle.$$

We use this fact in the following way. We call $c = (c_m)_{m\ odd}$ an odd normalised $(\mathbf{b^T},\mathbf{B^T})$**-boundary** if there is some even chain $e = \{e_m\}_{m\ even}$ with $c_m = b^T e_{m+1} + B^T e_{m-1}$ for all m. If we pair a normalised (b, B)-cocycle ϕ with a normalised (b^T, B^T)-boundary c we find

$$\langle \phi, c\rangle = \langle \phi, (b^T + B^T)e\rangle = \langle (b + B)\phi, e\rangle = 0.$$

There is an analogous definition in the case of even chains $c = (c_m)_{m\ even}$. All of the cocycles we consider in these notes are in fact defined as functionals on $\oplus_m \mathcal{A} \otimes \bar{\mathcal{A}}^{\otimes m}$. Henceforth we will drop the superscript on b^T, B^T and just write b, B for both boundary and coboundary operators as the meaning will be clear from the context.

We recall that the Chern character $Ch_*(u)$ of a unitary $u \in \mathcal{A}$ is the following (infinite) collection of odd chains $Ch_{2j+1}(u)$ satisfying $bCh_{2j+3}(u) + BCh_{2j+1}(u) = 0$,

$$Ch_{2j+1}(u) = (-1)^j j! u^* \otimes u \otimes u^* \otimes \cdots \otimes u \quad (2j + 2 \text{ entries}).$$

Exercise Check that $Ch_*(u)$ is a (b, B)-cycle.

Similarly, the (b, B) Chern character of a projection p in an algebra $\mathcal{A}$ is an even (b, B) cycle with $2m$-th term, $m \geq 1$, given by

$$Ch_{2m}(p) = (-1)^m \frac{(2m)!}{2(m!)}(2p-1) \otimes p^{\otimes 2m}.$$

For $m = 0$ the definition is $Ch_0(p) = p$.

Exercise Check that $Ch_*(p)$ is a (b, B)-cycle.

Since the (b, B) Chern character of a projection or unitary has infinitely many terms, we need some constraint on the cochains we pair them with.

If we allow only finitely supported cochains, then we obtain the usual cyclic cohomology groups $HC(\mathcal{A})$. The Chern character of a finitely summable spectral triple is finitely supported.

If we allow infinitely supported cochains which satisfy some decay condition α, then we get something we shall call $HC_\alpha(\mathcal{A})$. The most commonly used condition is to look at entire cochains, and the reason for this is that the JLO cocycle is entire; see [C1]. Very often one finds that for any reasonable decay condition α we have $HC_\alpha(\mathcal{A}) \cong HC(\mathcal{A})$, but general statements are hard to find.

A final warning: cyclic (co)homology of a C^*-algebra is trivial. It is necessary to work with a smooth subalgebra, or employ a fancy theory called local cyclic (co)homology due to Puschnigg. Alternatively, one could do KK with your favourite smooth algebras. This approach is developed by Cuntz.

In general I like the tension between continuous and smooth theories, and passing back and forth teaches you something about the way ‘differentiable structures’ appear.

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