Manifestly Supersymmetric Effective Actions on Walls and on Vortices

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§1. Introduction & Motivation

• brane world scenario

It is interesting and important to study the fat brane scenario in a case that a codimension is 1(2), that is, domain walls (vortices).

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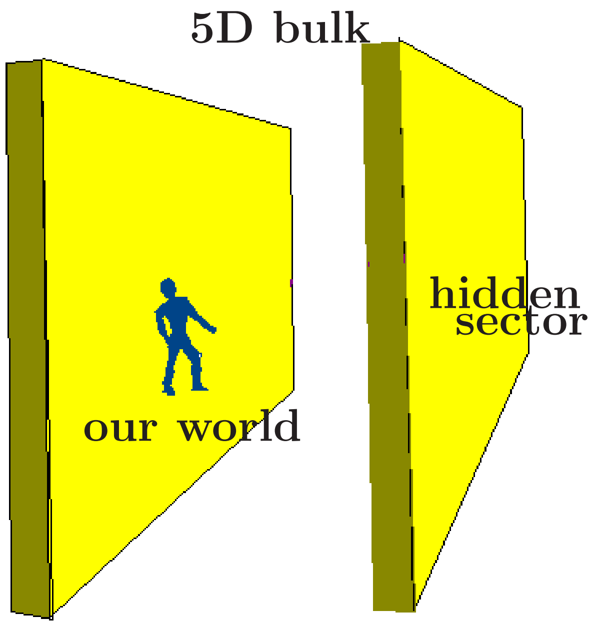

It is natural to consider domain walls (vortices) which are realized as 1/2 BPS states in a 5D(6D) SUSY theory. Therefore, it is important to investigate effective theories on domain walls (vortices), preserving the half super symmetry.

• Moduli Spaces

Moduli spaces for 1/2 BPS states in non-Abelian gauge theory were determined by

	codim.	
instantons	4	ADHM
monopoles	3	Nahm
vortices	2	Hanany-Tong
(domain-)walls	1	INOS

• Effective actions on walls and on vortices

We obtain formulas for effective actions on walls and on vortices in superfield formulation by Manton's method.

moduli parameters ϕ^α $\longrightarrow$ massless superfields on solitons $\phi^\alpha(x^\mu, \theta)$

Contents

§2 1/2 BPS Wall Solutions and Their Moduli Space

Phys.Rev.Lett 93(2004)161601[hep-th/0404198], hep-th/0405194

- **Our model:** 5D SUSY $U(N_\mathrm{C})$ gauge theory
  with $N_\mathrm{F}(> N_\mathrm{C})$ fundamental hypermultiplets

Field contents (bosonic part): $(M, N = 0, 1, 2, 3, 4)$

Vector multiplet: gauge field W_M, adjoint scalar Σ,

Hyper multiplets: complex $N_\mathrm{C} \times N_\mathrm{F}$ matrix $(H^i)^{rA} \equiv H^{irA}$,

$SU(2)_R$ $i = 1, 2$, color $r = 1, \cdots, N_\mathrm{C}$, flavor $A = 1, 2, \cdots N_\mathrm{F}$

Our Lagrangian (bosonic part)

$$\mathcal{L}\Big|_{\mathrm{bosonic}} = -\frac{1}{2g^2}\mathrm{Tr}[(F_{MN}(W))^2] + \frac{1}{g^2}\mathrm{Tr}[(\mathcal{D}_M\Sigma)^2] + (\mathcal{D}_M H)^\dagger_{iAr}\mathcal{D}^M H^{irA} - V_{\mathrm{pot}}$$

The scalar potential of this model

$$V_{\rm pot} = \frac{g^2}{4}{\rm Tr}\left[\left(c^a - (\sigma^a)^j{}_i H^i H^\dagger_j\right)^2\right] + H^\dagger_{iAr}[(\Sigma - m_A)^2]^r{}_s H^{isA}$$

Fayet-Illiopoulos parameter: $c_a = (0, 0, c > 0)$

non-degenerate masses m_A:

If $m_1 > m_2 > \cdots > m_{N_{\rm F}}$, then $SU(N_{\rm F}) \to U(1)^{N_{\rm F}-1}$

- color-flavor locking vacua

Vacua are labeled by $\langle A_1, A_2, \cdots, A_{N_{\rm C}}\rangle$

$$H^{1rA} = \sqrt{c}\,\delta^{A_r}{}_A, \qquad H^{2rA} = 0, \qquad \Sigma = {\rm diag}(m_{A_1}, \cdots, m_{A_{N_{\rm C}}})$$

$$\#{\rm vacua} = \frac{N_{\rm F}!}{N_{\rm C}!(N_{\rm F} - N_{\rm C})!}$$

where $U(1)^{N_{\rm F}-1} \to$ broken

For example, three vacua with $N_C = 2, N_F = 3$

vacuum $\langle 1, 2 \rangle$

$$H^1 = \sqrt{c} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$$

vacuum $\langle 1, 3 \rangle$

$$H^1 = \sqrt{c} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} m_1 & 0 \\ 0 & m_3 \end{pmatrix}$$

vacuum $\langle 2, 3 \rangle$

$$H^1 = \sqrt{c} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} m_2 & 0 \\ 0 & m_3 \end{pmatrix}$$

- Bogomol'nyi bound for walls

with boundaries $\langle A \rangle$ at $y = \infty$, and $\langle B \rangle$ at $y = -\infty$,

$$\begin{aligned} \mathcal{E} &= (\text{l.h.s of BPS eqs.})^2 + T_{\text{wall}} \\ &\geq T_{\text{wall}} = \int_{-\infty}^{\infty} dy \text{Tr}[\partial_y (c\Sigma)] = c \left(\sum_{r=1}^{N_C} m_{A_r} - \sum_{r=1}^{N_C} m_{B_r} \right) > 0 \end{aligned}$$

- 1/2 BPS equations for walls

We find a set of BPS equations: $(M)^A{}_B \equiv m_A \delta^A{}_B$

$$\begin{aligned} 0 &= \mathcal{D}_y H^1 + \Sigma H^1 - H^1 M \\ 0 &= \mathcal{D}_y \Sigma - \frac{g^2}{2}(c - H^1 H^{1\dagger}) \end{aligned}$$

we assume that solutions depend on only a coordinate $x^4 = y$, and for lorentz symmetry along the walls,

$$W_\mu = 0, (\mu = 0, 1, 2, 3).$$

• Solutions of the 1/2 BPS Eqs. for walls

Phys.Rev.Lett 93(2004)161601[hep-th/0404198], hep-th/0405194

$$\Sigma + iW_y \equiv S^{-1}\partial_y S, \qquad W_\mu = 0$$

$$H^1(y) = S^{-1}(y) H_0 e^{My}, \qquad H^2 = 0$$

with an arbitrary constant $N_{\rm C} \times N_{\rm F}$ matrix H_0, and an $S(y) \in GL(N_{\rm C}, C)$.

'Master equation' for a gauge invariant quantity $\Omega \equiv SS^\dagger$

$$\partial_y^2 \Omega - (\partial_y \Omega)\Omega^{-1}(\partial_y \Omega) = g^2 \left(c\Omega - H_0 e^{2My} H_0{}^\dagger \right)$$

Physical fields Σ, W_y, H^1 can be obtained by given H_0,

$$H_0 \longrightarrow \Omega(y) \longrightarrow S(y) \longrightarrow \Sigma,\ W_y,\ H^1$$

H_0 parametrize the moduli space for walls.

The simplest example with $N_{\rm C}=1$, $N_{\rm F}=2$ and $M=\mathrm{diag}(m,-m)$

A solution with $H_0=\sqrt{c}(1,1)$ in the strong coupling limit $g^2\to\infty$

$$
\begin{aligned}
\Sigma+iW_y &= m\tanh(2my)\\
H^1 &= S^{-1}H_0 e^{My}\\
&= \sqrt{c}\left(\frac{e^{my}}{\sqrt{\cosh(2my)}},\frac{e^{-my}}{\sqrt{\cosh(2my)}}\right)\\
&\to \begin{cases}\sqrt{c}(1,0)\ :\ \text{vacuum } \langle 1\rangle \text{ at } y\to\infty\\ \sqrt{c}(0,1)\ :\ \text{vacuum } \langle 2\rangle \text{ at } y\to-\infty\end{cases}
\end{aligned}
$$

• Total Moduli Space

The toatal moduli space of Walls is the deformed complex Grassmann manifold.

$$\mathcal{M}^{\rm total}_{\rm wall} = G_{\rm N_F, N_C} \simeq \frac{SU(N_{\rm F})}{SU(N_{\rm C} \times SU(N_{\rm F} - N_{\rm C}) \times U(1))}$$

$$\begin{aligned} \dim \mathcal{M}^{\rm total}_{\rm wall} &= 2N_{\rm C}(N_{\rm F} - N_{\rm C}) \\ &= \begin{cases} N_{\rm C}(N_{\rm F} - N_{\rm C}) & : \text{ positions of walls} \\ + N_{\rm F} - 1 & : \quad \text{NG modes} \\ + (N_{\rm C} - 1)(N_{\rm F} - N_{\rm C} - 1) & : \quad \text{QNG modes} \end{cases} \end{aligned}$$

Let us promote

moduli parameters ϕ^α $\longrightarrow$ massless superfields on the walls $\phi^\alpha(x^\mu, \theta)$

and obtain an effective acton on the walls.

§3. Manifestly Supersymmetric Effective Action on (Multi-) Walls

hep-th/0502***

To obtain the effective action with manifest supersymmetry, let us consider superfield formulation respecting the unbroken half supersymmetry on the BPS walls.

superfield respecting configurations for walls

$$
\begin{aligned}
\text{Hypermultiplet} \longrightarrow \ &\text{chiral}: \ \hat{H}^1(x,\theta)\big|_{\theta=0} = H^1(x), \\
&\text{chiral}: \ \hat{H}^2(x,\theta)\big|_{\theta=0} = H^2(x) \\
\text{5D vector multiplet} \longrightarrow \ &\text{chiral}: \ \hat{\Sigma}(x,\theta)\big|_{\theta=0} = \Sigma(x) + iW_y(x), \\
&\text{vector}: \ \hat{V}(x,\theta,\bar{\theta})\big|_{\bar{\theta}\gamma_\mu\theta} = W_\mu(x), \quad \text{(WZ gauge)}
\end{aligned}
$$

5D Action in superfield formulation   A.Hebecker Nucl. Phys. B 632, 101 (2002)

$$\begin{aligned}\mathcal{L}_{\rm w} &= \int dy \mathcal{L} \\ &= -T_{\rm wall} \\ &\quad + \int dy d^4\theta \,{\rm Tr}\left[\frac{1}{2g^2}(e^{-2\hat{V}}\hat{D}_y e^{2\hat{V}})^2 + 2c\hat{V}\right] \\ &\quad + \int dy d^4\theta \,{\rm Tr}\left[\hat{H}^{\dagger 1} e^{-2\hat{V}} \hat{H}^1 + \hat{H}^{\dagger 2} e^{2\hat{V}} \hat{H}^2\right] \\ &\quad + \int dy d^2\theta \left[\frac{1}{4g^2}\hat{W}^\alpha \hat{W}_\alpha + \hat{H}^{2\dagger}\left(\hat{D}_y \hat{H}^1 - \hat{H}^1 M\right)\right] + {\rm c.c.}\end{aligned}$$

where

$$T_{\rm wall} = [{\rm Tr}(c\Sigma)]^{\infty}_{-\infty}$$

covariant derivatives

$$\begin{aligned}\hat{D}_y e^{2\hat{V}} &= \partial_y e^{2\hat{V}} + \hat{\Sigma} e^{2\hat{V}} + e^{2\hat{V}}\hat{\Sigma} \\ \hat{D}_y \hat{H}^1 &= \partial_y \hat{H} + \hat{\Sigma}\hat{H}^1\end{aligned}$$

Manton's Method (slow moving approximation)

$$\partial_y \phi = \mathcal{O}(1)\phi, \quad \partial_\mu \phi = \mathcal{O}(\lambda)\phi, \quad \lambda \ll 1, \quad \mu = 0,1,2,3$$

$\Rightarrow$ For consistency with SUSY, we have to take rules,

$$d\theta \sim \frac{\partial}{\partial\theta} \sim \mathcal{O}(\lambda^{\frac{1}{2}})$$

By use of these rules, we can set ansatz for wall configularations cosistently.

$$\hat{H}^1 \sim \mathcal{O}(1), \quad \hat{H}^2 \sim \mathcal{O}(\lambda)$$
$$\hat{\Sigma} \sim \mathcal{O}(1), \quad \hat{V} \sim \mathcal{O}(1), \quad \left(W_\mu \sim \mathcal{O}(\lambda)\right)$$

$$\Rightarrow \int dy d^2\theta \left[\frac{1}{4g^2}\hat{W}^\alpha \hat{W}_\alpha\right] \sim \mathcal{O}(\lambda^4), \quad \int dy d^4\theta \mathrm{Tr}\left[\hat{H}^{\dagger 2} e^{2\hat{V}} \hat{H}^2\right] \sim \mathcal{O}(\lambda^4)$$

Omitting $\mathcal{O}(\lambda^4)$ terms,

$\Leftrightarrow$ $N = 2$ theory is broken into $N = 1$

$$
\begin{aligned}
\mathcal{L}_{\rm w} &= -T_{\rm wall} \\
&+ \int dy d^4\theta \mathrm{Tr}\left[\frac{1}{2g^2}(e^{-2\hat{V}}\hat{D}_y e^{2\hat{V}})^2 + 2c\hat{V} + \hat{H}^{\dagger 1} e^{-2\hat{V}}\hat{H}^1\right] \\
&+ \int dy d^2\theta \left[\hat{H}^{2\dagger}\left(\hat{D}_y\hat{H}^1 - \hat{H}^1 M\right)\right] + \mathrm{c.c.}
\end{aligned}
$$

Equations of motion for auxiliary fields $\hat{V}, \hat{H}^2$,

$$
\begin{aligned}
\hat{D}_y(e^{-2\hat{V}}\hat{D}_y e^{2\hat{V}}) &= g^2\left(c - e^{-2\hat{V}}\hat{H}^1\hat{H}^{1\dagger}\right) \\
\hat{D}_y\hat{H}^1 &= \hat{H}^1 M
\end{aligned}
$$

- the lowest components of these Eqs. → 1/2 BPS equations for walls
- higher components of these Eqs.→ equations for y-dependence of higher components

All components of these equations are solved with a chiral fields $\hat{S}$ by

$$
\begin{aligned}
\hat{\Sigma} &= \hat{S}^{-1}\partial_y\hat{S}, \\
\hat{H}^1 &= \hat{S}^{-1}\hat{H}_0 e^{My}
\end{aligned}
$$

$\hat{H}_0$: y-independent chiral fields

and the vector field $\hat{\Omega} \equiv \hat{S} e^{2\hat{V}} \hat{S}^\dagger$ are determined by
supersymmetric master equations

$$\partial_y (\hat{\Omega}^{-1} \partial_y \hat{\Omega}) = g^2 (c - \hat{\Omega}^{-1} \hat{H}_0 e^{2My} \hat{H}_0)$$

Solutions are obtained by use of the solution of the bosonic master eq.

$$\Omega = \Omega_{\rm sol}(H_0, H_0^\dagger) \quad \rightarrow \quad \hat{\Omega} = \Omega_{\rm sol}(\hat{H}_0, \hat{H}_0^\dagger)$$

By substituting these solution, we obtain

$$\mathcal{L}_{\rm w} = -T_{\rm wall} + \int d^4\theta K_{\rm wall} + \mathcal{O}(\lambda^4)$$

which turns out to be an effective action on the walls.
Kähler potential of the effective action is given by,

$$K_{\rm wall} = \int dy {\rm Tr} \underbrace{\left[\frac{1}{2g^2} (\hat{\Omega}^{-1} \partial_y \hat{\Omega})^2 + c \log \hat{\Omega} + \hat{\Omega}^{-1} \hat{H}_0 e^{2My} \hat{H}_0^\dagger \right]}_{\text{Lagrangian for } \hat{\Omega} \text{ with a source } \hat{H}_0 e^{2My} \hat{H}_0^\dagger} \Bigg|_{\hat{\Omega} = \hat{\Omega}_{\rm sol}}$$

- Example with $SU(N)_{\rm F} \times SU(N)_{\rm F'}$, $\quad (N_{\rm F} = 2N_{\rm C} \equiv 2N)$

Hypermultiplets: $H^i = (H^i_+, H^i_-)$

	$U(N)_{\rm C}$	$SU(N)_{\rm F}$	$SU(N)_{\rm F'}$	mass
H^i_+	N	$\bar{N}$	1	$\frac{m}{2}$
H^i_-	N	1	$\bar{N}$	$-\frac{m}{2}$

A moduli matrix for N-walls solution is

$$H_0 = \sqrt{c}(1_N,\, e^{\phi})$$

where a moduli parameter ϕ is an complex $N \times N$ matrix.

$$\Downarrow \quad \phi \longrightarrow \phi(x,\theta)\text{: chiral field}$$

Kähler potential of the effective action for arbitrary g:

$$K_{\rm wall} = \frac{c}{4m}{\rm Tr}\left[\left(\log(e^{\phi}e^{\phi^\dagger})\right)^2\right] + \mathcal{O}(\lambda^2)$$

We believe that this gives Skyrm model in superfield formulation.

§4. Effective Action on Vortices

- 6D $N = 1$(8 SUSY) theory ($M = 0$) in superfield formulation

N. Arkani-Hamed, T. Gregoire and J. Wacker, JHEP 0203, 055 (2002)

$\Downarrow$ Neglecting halves of $N = 2$ supermultiplets

- 4D $N = 1$(4 SUSY) effective theory on BPS vortices

$$\mathcal{L}_{\rm v} = \underbrace{-2\pi c\, k}_{\text{tension of } k \text{ vortices}} + \int d^4\theta K_{\rm vortex} + \mathcal{O}(\lambda^4)$$

Kähler potential of the effective action,

$$K_{\rm vortex} = \frac{1}{2i}\int dz dz^* \mathcal{L}_\Omega \Big|_{\hat{\Omega}=\hat{\Omega}_{\rm sol}}$$

$$\mathcal{L}_\Omega = {\rm Tr}\left[\frac{2}{g^2}(\hat{\Omega}^{-1}\partial\hat{\Omega})(\hat{\Omega}^{-1}\bar{\partial}\hat{\Omega}) + c\log\hat{\Omega} + \hat{\Omega}^{-1}H_0H_0^\dagger\right] + \mathcal{L}_{\rm WZW}$$

with a Wess-Zumino-Witten term

$$\mathcal{L}_{\rm WZW} = \frac{4}{g^2}{\rm Tr}\left[\bar{\partial}\Phi\frac{\sinh L_\Phi - L_\Phi}{L_\Phi^2}\partial\Phi\right],$$

where

$$\Phi \equiv \log\hat{\Omega}, \quad L_\Phi X = [\Phi, X]$$

§5. Summary and Discussion

- We obtain formulas of effective actions on walls and on vortices in superfield formulation

- Neglecting halves of $N = 2$ supermultiplets consistently
  $=$ Obtaining an effective action on a 1/2 BPS state

- Kähler potentials for effective actions are obtained by Lagrangians which give supersymmetric master equations of Ω as equations of motions.

There are many future problem.

- Quantum corrections

- Generalization: non-minimal kinetic term, SUGRA, adjoint scalars, other gauge group,. . .

- Localization of gauge fields

- SUSY breaking

- Method to construct the effective actions without exact solutions

- Investigation of solutions in the case of $g^2 < \infty$

- . . .