2

FROM LAGRANGIAN TO CROSS SECTION

Lagrangian

$\downarrow$

Feynman rule

$\downarrow$

Feynman graph

$\downarrow$

Scattering amplitude (T matrix element)

$\downarrow$

Cross section / Decay width

+ Parton model

vacuum

$$\langle 0|0\rangle = 1 \qquad\qquad |0\rangle : \text{ dim } 0$$

one-particle state

$$\langle p|p'\rangle = 2p^0 (2\pi)^3 \delta^3(p-p') \qquad\qquad |p\rangle : \text{ dim } -1$$

relativistic normalization

$$1 = |0\rangle\langle 0| + \sum_p |p\rangle\langle p| + \cdots$$

$$\sum_p = \frac{d^3p}{(2\pi)^3 2p^0} \quad : \text{ dim } 2$$

↳ phase space

▸ We adopt this normalization for fermions (as well as bosons).

cf. Bjorken-Drell has

$$\sum_p = \frac{m\, d^3p}{(2\pi)^3 p^0}$$

which is convenient for massless limit.

$$\varphi(x) = \sum_p \left[a(p)\, e^{-ip\cdot x} + a^\dagger(p)\, e^{+ip\cdot x} \right]$$

(real)

$$|p\rangle = a^\dagger(p)|0\rangle$$

$$\psi(x) = \sum_p \sum_h \left[a_h(p)\, u_h(p)\, e^{-ip\cdot x} + b_h^\dagger(p)\, v_h(p)\, e^{+ip\cdot x} \right]$$

(Dirac)

h : helicity (or spin)

$$\bar{u}_h(p)\, u_{h'}(p) = 2m\, \delta_{hh'}$$

$$\bar{v}_h(p)\, v_{h'}(p) = -2m\, \delta_{hh'}$$

$$\bar{u}_h(p)\, v_{h'}(p) = 0$$

S matrix

$$S_{fi} = \langle f\ \mathrm{out} | i\ \mathrm{in} \rangle$$

$$S_{fi} = 1_{fi} + i\,(2\pi)^4 \delta^4(P_f - P_i)\, T_{fi}$$

注意：Tの符号は人によって異なる.

This alone does not fix the normalization. Need a convention for 1_{fi}

$$1 = |0\rangle\langle 0| + \sum_p |p\rangle\langle p| + \cdots \qquad \sum_p = \frac{d^3p}{(2\pi)^3 2p^0}$$

Unitarity

$$SS^\dagger = S^\dagger S = 1$$

In terms of T

$$S = 1 + iT \qquad S^\dagger = 1 - iT^\dagger$$

$$S^\dagger S = 1 + i(T - T^\dagger) + T^\dagger T = 1$$

$$\Rightarrow \quad -i(T - T^\dagger) = T^\dagger T$$

$$-i\left(T_{fi} - T^\dagger_{fi}\right) = \sum_n (2\pi)^4 \delta^4(P_n - P_i)\, T^\dagger_{fn} T_{ni}$$

$$= \int d\Phi_n\, T^\dagger_{fn} T_{ni}$$

前方散乱 $f = i$ (elastic)

$$-i\left(T_{ii} - T^\dagger_{ii}\right) = 2\,\mathrm{Im}\, T_{ii}$$

$$\int d\Phi_n\, T^\dagger_{in} T_{ni} = \int d\Phi_n\, |T_{ni}|^2 = 2s\bar{\beta}_i \sum_n \sigma(i \to n)$$

(For i = 2-particle state)

$$\Rightarrow \quad \boxed{\mathrm{Im}\, T_{ii} = s\bar{\beta}_i\, \sigma_{i,\mathrm{tot}}} \qquad \text{(Optical Theorem)}$$

$$= 2\sqrt{s}\, |\mathbf{k}|_{\mathrm{cm}}\, \sigma_{i,\mathrm{tot}}$$

UNITARITY AND UNITARITY LIMIT (TWO-BODY SCATTERING)

$$S^{\dagger} S = 1 \tag{1}$$

EASIER TO WORK WITH A BASIS IN WHICH S IS (CLOSER TO) DIAGONAL

→ PARTIAL WAVE

(J IS CONSERVED → NO ~~OFF-DIAGONAL~~ ELEMENTS)

$$S = \sum_J S^{(J)} \tag{2}$$

$$S^{(J)} = 1 + i\, T^{(J)} \tag{3}$$

UNITARITY IN TERMS OF $T^{(J)}$

$$-i \left(T^{(J)}_{fi} - T^{(J)*}_{if} \right) = \sum_n T^{(J)*}_{nf} \, T^{(J)}_{ni} \qquad i, f, n : \tag{4}$$

$f = i$

$$2 \, \mathrm{Im}\, T^{(J)}_{ii} = \sum_n \left| T^{(J)}_{ni} \right|^2$$

$$= \left| T^{(J)}_{ii} \right|^2 + \sum_{n \neq i} \left| T^{(J)}_{ni} \right|^2$$

$$\geq \left| T^{(J)}_{ii} \right|^2 \tag{5}$$

On the other hand

$$\left| T^{(J)}_{ii} \right| \geq \mathrm{Im}\, T^{(J)}_{ii} \tag{6}$$

These two $\overset{\text{via}}{}$ eqs give

$$\left| T^{(J)}_{ii} \right| \leq 2 \qquad \text{(LIMIT FOR \underline{ELASTIC} CHANNEL)} \tag{7}$$

INELASTIC CHANNEL $f \neq i$

$$|T_{fi}^{(J)}|^2 \leq \sum_{n \neq i} |T_{ni}^{(J)}|^2$$

$$= 2\, \mathrm{Im}\, T_{ii}^{(J)} - |T_{ii}^{(J)}|^2 \qquad \text{(UNITARITY)}$$

$$\leq 2|T_{ii}^{(J)}| - |T_{ii}^{(J)}|^2 \qquad \text{(Eq. (6))}$$

$$= |T_{ii}^{(J)}| \left(2 - |T_{ii}^{(J)}|\right)$$

$$\leq 1 \qquad \text{(Eq. (7))} \qquad (8)$$

INVARIANT AMPLITUDE (MASSLESS TWO-BODY SCATTERING)
SPINLESS PARTICLES. (OR $\lambda_i = \lambda_f = 0$ IN GENERAL)

$$\mathcal{M}_{fi} = 8\pi \sum_{J=0}^{\infty} (2J+1)\, T_{fi}^{(J)}\, P_J(\cos\theta) \qquad (9)$$

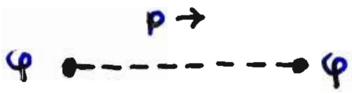
Lagrangian → Feynman rule

$$\begin{cases} \text{Free part (or kinetic part)} & \rightarrow \quad \text{Propagator} \\ \text{Interaction} & \rightarrow \quad \text{Vertex} \end{cases}$$

- It is sometimes useful to include masses in interactions

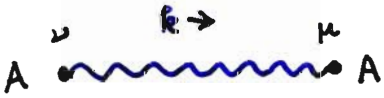
Propagator

"universal"



$$\frac{i}{p^2 - m^2 + i0}$$

$$\frac{i}{\not{p} - m + i0} = \frac{i(\not{p} + m)}{p^2 - m^2 + i0}$$



massless

$$\frac{i}{k^2 + i0}\left(-g_{\mu\nu} + (1-\alpha)\frac{k_\mu k_\nu}{k^2}\right)$$

$$-g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$\alpha = 1$: Feynman gauge $\quad \dfrac{-ig_{\mu\nu}}{k^2 + i0}$

$\alpha = 0$: Landau gauge

massive (unitary gauge)

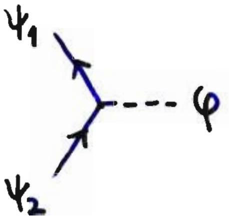
$$\frac{i\left(-g_{\mu\nu} + \frac{k_\mu k_\nu}{M^2}\right)}{k^2 - M^2 + i0}$$

Vertex

from $i\mathcal{L}_{int}$

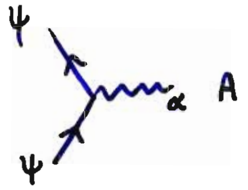
Derivative の無い場合

(例1) $\mathcal{L} = if\bar{\psi}_1\gamma_5\psi_2\varphi$



$i \cdot if\gamma_5 = -f\gamma_5$

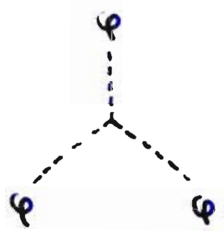
(例2) $\mathcal{L} = -e\bar{\psi}\gamma_\mu\psi A^\mu$



$i(-e)\gamma_\mu \cdot g^{\mu\alpha} = -ie\gamma_\alpha$

(例3) 同種の場が複数ある場合 → symmetry factor

$\mathcal{L} = \mu\varphi^3 = 6\mu\cdot\left(\frac{1}{3!}\varphi^3\right)$



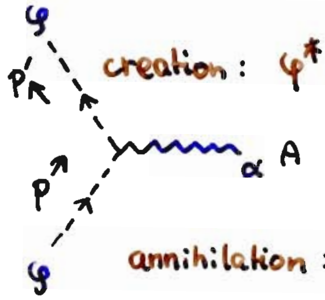
$6i\mu$

- This prescription is to minimize the appearance of symmetry factors in the Feynman graph calculation. It is always safe to use Wick's theorem when there is a loop (or when confused).

Derivative のある場合

微分 → momentum

(例4) $\mathcal{L} = -ie\,(\varphi^* \partial_\mu \varphi - \partial_\mu \varphi^* \varphi)\, A_\mu$



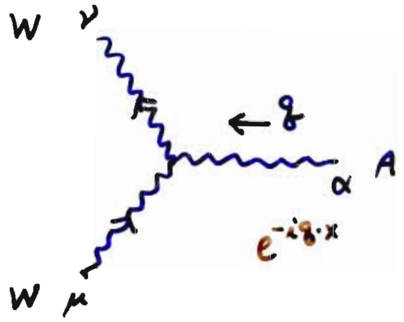
creation : $\varphi^* \sim e^{+ip'\cdot x}$

annihilation : $\varphi \sim e^{-ip\cdot x}$

$$i\cdot(-ie)\cdot[(-ip_\alpha) - (ip'_\alpha)] = -ie(p+p')_\alpha$$

(例5) $\mathcal{L} = -ie\kappa\, W_\mu^\dagger W_\nu F^{\mu\nu}$ $\quad (F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu)$

$$= -ie\kappa\, W_\mu^\dagger W_\nu (\partial^\mu g^{\nu\alpha} - \partial^\nu g^{\mu\alpha}) A_\alpha$$



"anomalous" magnetic moment of W

$$i(-ie\kappa)[(-iq_\mu) g_{\nu\alpha} - (-iq_\nu) g_{\mu\alpha}]$$

$$= -ie\kappa(q_\mu g_{\nu\alpha} - q_\nu g_{\mu\alpha})$$

Feynman rule → Feynman graph

Lagrangian ↕ Feynman rule → Feynman graph ↕ scattering amplitude (S matrix element)

Feynman graph → Green 関数 → scattering amplitude

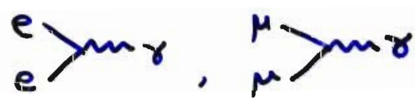
Feynman rule
- propagator
- vertex

Draw all possible graphs for a given process using the vertices (and propagators) of the model.

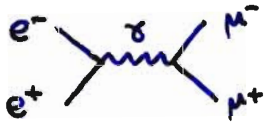
For a given process, perturbation expansion ≈ loop expansion

(例) $e^+e^- \to \mu^+\mu^-$ in QED

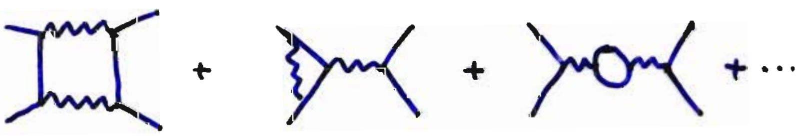
vertices :



Tree $O(e^2)$



One-loop $O(e^4)$



However, in a multi-coupling theory with drastically different couplings One-loop ≫ Tree possible

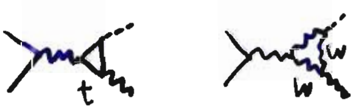
(例) $e^+e^- \to H\gamma$

Tree

$O(eg\underbrace{\frac{m_e}{m_W}}_{\text{Yukawa}})$

$\hat{}$

1-loop



$O(e^3 g)$

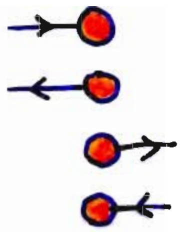
Feynman graph → Scattering amplitude

Vertex	→	Vertex factor
Internal line	→	Propagator
Loop	→	$\int \frac{d^4k}{(2\pi)^4}$
External line	→	Wave fn

scalar → 1

fermion

initial	粒子	$u(p)$	
	反粒子	$\bar{v}(p)$	
final	粒子	$\bar{u}(p)$	
	反粒子	$v(p)$	



vector

initial	ϵ_μ
final	ϵ_μ^*

Fermion に関する注意.

Closed fermion loop → factor (-1)

fermion lines を入れかえて得られる graph は相対符号 −

$$\underbrace{a^\dagger(1)a^\dagger(2)}_{\text{fermion}}|0\rangle = \underbrace{-}_{\text{ambiguity}}a^\dagger(2)a^\dagger(1)|0\rangle$$

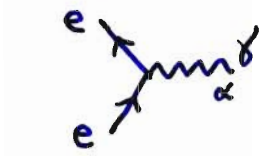
以上の規則は 以前定義した iT_{fi} を与える.

$$iT_{fi} \equiv i\mathcal{M}$$

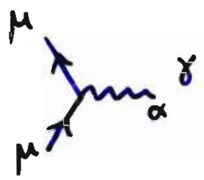
(例) $e^+e^- \to \mu^+\mu^-$ (QED)

$e^-(k) + e^+(\bar{k}) \to \mu^-(p) + \mu^+(\bar{p})$

Vertices

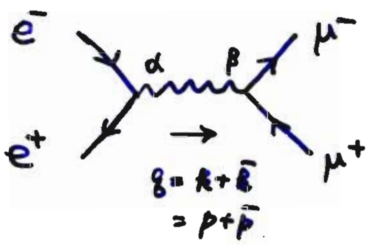


$+ie\gamma_\alpha$



同

initial ⟶ final

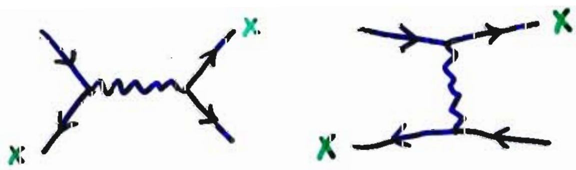


(Feynman gauge)

$$i\mathcal{M} = \bar{u}(p)\, ie\gamma_\beta\, v(\bar{p})\, \frac{-ig^{\alpha\beta}}{q^2}\, \bar{v}(\bar{k})\, ie\gamma_\alpha\, u(k)$$

$$\mathcal{M} = \frac{e^2}{q^2}\, \bar{u}(p)\gamma_\alpha v(\bar{p})\; \bar{v}(\bar{k})\gamma^\alpha u(k)$$

(例) Bhabha 散乱 $e^+e^- \to e^+e^-$ (QED)



この2つのグラフは X のついた外線の入れかえで互いに移るので
相対符号(−1)が必要

Scattering amplitude → Cross section / Width

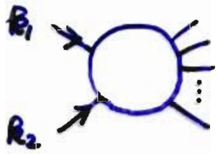
Decay rate (in rest frame)

$$d\Gamma = \frac{1}{2M} \bar{\sum_{\text{spin etc.}}} |\mathcal{M}|^2 \, d\Phi$$

↳ phase space

General frame : $\frac{1}{2M} \to \frac{1}{2P^0}$

Scattering cross section



$$d\sigma = \frac{1}{2s\bar{\beta}_i} \bar{\sum} |\mathcal{M}|^2 \, d\Phi$$

$\bar{\sum}$: average for initial state
sum for final state

$s = E_{cm}^2$

$\bar{\beta}_i = \frac{2|\mathbf{k}|}{\sqrt{s}} = \frac{|\mathbf{k}|}{|\mathbf{k}|}$ for massless initial particles

or

$s\bar{\beta}_i = \sqrt{(k_1 \cdot k_2)^2 - M_1^2 M_2^2}$

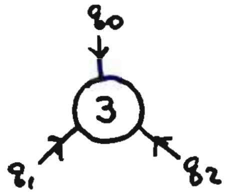
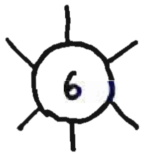
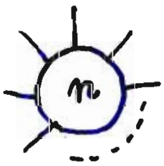
or

$$\bar{\beta}_i^2 = \left[1 - \frac{(M_1+M_2)^2}{s}\right]\left[1 - \frac{(M_1-M_2)^2}{s}\right]$$

$$= \frac{1}{s^2}\left(s^2 + M_1^4 + M_2^4 - 2sM_1^2 - 2sM_2^2 - 2M_1^2M_2^2\right)$$

$$= \frac{1}{s^2}\left(\sqrt{s}+M_1+M_2\right)\left(\sqrt{s}+M_1-M_2\right)\left(\sqrt{s}-M_1+M_2\right)\left(\sqrt{s}-M_1-M_2\right)$$

Independent invariants

	with	invariants	# incl. "masses"	# excl. masses
	$q_0+q_1=0$	q_1^2	1	-1 (1 constraint)
	$q_0+q_1+q_2=0$	$\oplus\ q_2^2,\ q_1\cdot q_2$	3	0
		$\oplus\ q_3^2$, $q_1\cdot q_3,\ q_2\cdot q_3$	6	2 (s, t)
		$\oplus\ q_4^2,\ q_1\cdot q_4$, $q_2\cdot q_4,\ q_3\cdot q_4$	10	5
		$\oplus\ q_5^2,\ q_1\cdot q_5,\ q_2\cdot q_5$, $q_3\cdot q_5,\ q_4\cdot q_5$	14	8
	$\sum_{i=0}^{n-1} q_i = 0$	$\oplus\ 4$	$4n-\underbrace{10}_{\text{\# generators of Poincaré group}}$	$3n-10$

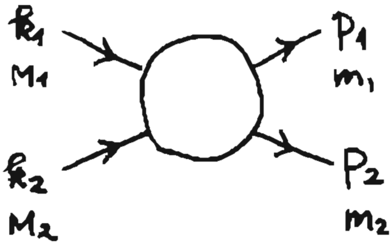
But q_5 has only 4 components
→ 4 invariants out of 5 independent

Physical processes: initial momentum configuration fixed

assume: spin summed

	$1\to2$	0		
	$1\to3$	2		
	$2\to2$	$2-1=1$	($\cos\theta$ or t)	E_{cm} fixed
	$2\to3$	$5-1=4$		

$2 \to 2$ reactions (Kinematics)



$$k_1 + k_2 = p_1 + p_2$$

$$s = (k_1+k_2)^2 = (p_1+p_2)^2 = E_{cm}^2$$

$$t = (k_1-p_1)^2 = (k_2-p_2)^2$$

$$u = (k_1-p_2)^2 = (k_2-p_1)^2$$

$$s + t + u = \sum (\text{mass})^2$$
$$= M_1^2 + M_2^2 + m_1^2 + m_2^2$$

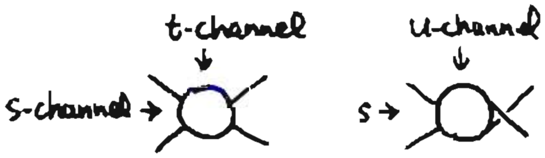
For massless particles

$$k_1 = E(1\ 0\ 0\ 1) \qquad p_1 = E(1\ \sin\theta\ 0\ \cos\theta)$$
$$k_2 = E(1\ 0\ 0\ {-1}) \qquad p_2 = E(1\ {-\sin\theta}\ 0\ {-\cos\theta})$$

$$s = 4E^2 \qquad t = -2E^2(1-\cos\theta) = -s\,\frac{1-\cos\theta}{2} \qquad u = -2E^2(1+\cos\theta) = -s\,\frac{1+\cos\theta}{2}$$

$s+t+u=0$ $\qquad$ $t=0$：前方散乱 $\cos\theta = 1$ $\qquad$ $u=0$：後方散乱 $\cos\theta = -1$

Crossed channels



Tree $2 \to 2$ graphs : Possible topology

s-channel exchange $\qquad$ t-channel $\qquad$ u-channel $\qquad$ contact

Phase space

$$d\Phi = (2\pi)^4 \delta^4\left(q - \sum_{i=1}^{n} p_i\right) \prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3 2p_i^0}$$

dim $-4+2n$

$q = k_1 + k_2$ for scattering

$p_i^0 = \sqrt{\vec{p}_i^{\,2} + m_i^2}$

◇ Lorentz invariant

because

$$\frac{d^3 p_i}{2p_i^0} = d^4 p_i \, \delta(p_i^2 - m_i^2)\, \theta(p_i^0)$$

take the same normalization for fermions and bosons ($\bar{u}u = 2m$)

Note : if there are identical particles, divide by the symmetry factor $n!$ (for total cross section)

1-body phase space

$3-4 = -1$ variables !

$$d\Phi_1 = 2\pi \, \delta^4(q - p_1) \frac{d^3 p_1}{2p_1^0}$$

$$= 2\pi \, \delta^4(q - p_1) \, d^4 p_1 \, \delta(p_1^2 - m_1^2)\, \theta(p_1^0)$$

$$= 2\pi \, \delta(q^2 - m_1^2)$$

resonance formation.

Ecm constrained

(例) $e\bar{\nu} \to W$

$$|\mathcal{M}|^2 = g^2 m_W^2 = \frac{4\pi\alpha}{\sin^2\theta_W} m_W^2 \qquad \text{(後出)}$$

$$\sigma(e\bar{\nu} \to W) = \frac{1}{2s} \, \frac{1}{2} |\mathcal{M}|^2 \cdot 2\pi\delta(s - m_W^2) = \frac{2\pi^2\alpha}{\sin^2\theta_W} \delta(s - m_W^2)$$

$$\sigma(u\bar{d} \to W) = \frac{\pi^2\alpha}{3\sin^2\theta_W} \delta(s - m_W^2)$$

unitarity → Breit-Wigner shape

$$\frac{1}{(s - m_W^2)^2 + m_W^2 \Gamma_W^2} \sim \frac{\pi}{m_W \Gamma_W} \delta(s - m_W^2)$$

2-body phase space

6-4=2 variables

Constraints for p_i's

$p_1^2 = m_1^2, \quad p_2^2 = m_2^2$

$$d\Phi_2 = \frac{1}{(2\pi)^2}\,\delta^4(q-p_1-p_2)\,\frac{d^3p_1}{2p_1^0}\,\underline{\frac{d^3p_2}{2p_2^0}}$$

$$= \frac{1}{(2\pi)^2}\,\underline{\delta^4(q-p_1-p_2)}\,\frac{d^3p_1}{2p_1^0}\,\underline{d^4p_2}\,\delta(p_2^2-m_2^2)\,\theta(p_2^0) \qquad p_1^2 = m_1^2$$

$$= \frac{1}{(2\pi)^2}\,\frac{d^3p_1}{2p_1^0}\,\delta((q-p_1)^2-m_2^2)\,\theta(q^0-p_1^0) \qquad p_1^2 = m_1^2, \quad p_2 = q - p_1$$

$$d^3p_1 = p_1^2\,dp_1\,d\Omega_1 \qquad (p_1 = |\mathbf{p}_1|)$$

$$(p_1^0)^2 = p_1^2 + m_1^2$$

$$p_1^0\,dp_1^0 = p_1\,dp_1$$

$$= p_1\,p_1^0\,dp_1^0\,d\Omega_1$$

c.m. frame

$$q = (\sqrt{s}, 0, 0, 0)$$

$$(q-p_1)^2 = q^2 - 2q\cdot p_1 + p_1^2 = s - 2\sqrt{s}\,p_1^0 + m_1^2$$

$$\delta((q-p_1)^2 - m_2^2) = \delta(s + m_1^2 - m_2^2 - 2\sqrt{s}\,p_1^0)$$

$$= \frac{1}{2\sqrt{s}}\,\delta\left(p_1^0 - \frac{s+m_1^2-m_2^2}{2\sqrt{s}}\right)$$

$$= \frac{1}{(2\pi)^2}\,\frac{1}{2}\,p_1\,dp_1^0\,d\Omega_1\,\frac{1}{2\sqrt{s}}\,\delta(p_1^0 - \cdots)\,\theta(\sqrt{s} - p_1^0)$$

$$p_1 = \sqrt{(p_1^0)^2 - m_1^2} \equiv \frac{\sqrt{s}}{2}\,\bar{\beta}_f$$

$$= \frac{\bar{\beta}_f}{32\pi^2}\,d\Omega_1$$

$$\left(= \frac{\bar{\beta}_f}{16\pi}\,d\cos\theta_1 \right)$$

$$\bar{\beta}_f = \frac{1}{s^2}\left[s^2 - 2(m_1^2+m_2^2)s + (m_1^2-m_2^2)^2\right]$$

$$\frac{dt}{8\pi s \bar{\beta}_i}$$

$$p_1^0 = \frac{s+m_1^2-m_2^2}{2\sqrt{s}}$$

$$p_2^0 = \frac{s-m_1^2+m_2^2}{2\sqrt{s}}$$

$$|\mathbf{p}_1| = |\mathbf{p}_2| = \frac{\sqrt{s}}{2}\,\bar{\beta}_f$$

$$\cos\theta_2 = -\cos\theta_1$$

$$\phi_2 = \phi_1 + \pi$$

both massless $\rightarrow$ $\bar{\beta}_f = 1$

equal mass $\rightarrow$ $\bar{\beta}_f = \beta_f = \sqrt{1 - \frac{4m^2}{s}}$

one massless $\rightarrow$ $\bar{\beta}_f = 1 - \frac{m^2}{s}$

3-body phase space

$$d\Phi_3 = \frac{1}{(2\pi)^5}\delta^4(q-p_1-p_2-p_3)\frac{d^3p_1}{2p_1^0}\frac{d^3p_2}{2p_2^0}\frac{d^3p_3}{2p_3^0}$$

$$= \frac{1}{(2\pi)^5}\frac{d^3p_1}{2p_1^0}\frac{d^3p_2}{2p_2^0}\delta((q-p_1-p_2)^2-m_3^2)\,\theta(\cdots)$$

$$= \frac{1}{4(2\pi)^5}\,p_1dp_1^0d\Omega_1\,p_2dp_2^0d\Omega_2\,\delta(q^2-2q\cdot(p_1+p_2)+2p_1\cdot p_2+m_1^2+m_2^2-m_3^2)$$

c.m. frame

$$= \frac{1}{4(2\pi)^5}\,p_1p_2\,dp_1^0dp_2^0d\Omega_1\,d\cos\theta_{12}\,d\phi_{12}\,\delta(-2p_1p_2\cos\theta_{12}+\cdots)$$

$$= \frac{1}{8(2\pi)^5}\,dp_1^0dp_2^0d\Omega_1 d\phi_{12}$$

$$= \frac{q^2}{1024\pi^5}\,dz_1dz_2d\Omega_1\,d\phi_{12}$$

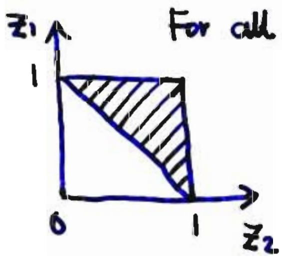
$$\begin{cases} z_1 = \dfrac{2q\cdot p_1}{q^2} = \dfrac{2p_1^0}{\sqrt{q^2}}\Big|_{cm} \\[2ex] z_2 = \dfrac{2q\cdot p_2}{q^2} \end{cases}$$

(Normalized energy)

Decay of an unpolarized particle

$\rightarrow\ d\Omega_1 d\phi_{12} = 8\pi^2$

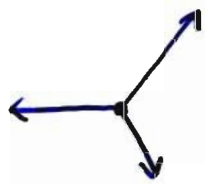
$$= \frac{q^2}{128\pi^3}\,dz_1\,dz_2 \qquad (*)$$



For all massless final states

$0 \le z_1, z_2 \le 1$

$z_1 + z_2 \ge 1$



3体 decay の configuration は 2粒子の energy だけで完全に決まる。

$z_3 = \dfrac{2q\cdot p_3}{q^2}$ を定義すると $z_1+z_2+z_3 = 2$.

上の表式(*)で dz_1dz_2 のかわりに dz_1dz_3 または dz_2dz_3 としてもよい。

$\boxed{\text{宿題}}$ 1) $p_1^2 = m^2$, $p_2^2 = p_3^2 = 0$ のとき $\{z_i\}$ の boundary を求めよ。

2) 一般の mass のとき 〃 .

Spin sum をとる場合の $\Sigma|\mathcal{M}|^2$ の計算法

フェルミオンの取扱 --- Trace technique

$$\sum_{\text{spins}} \left| \bar{u}(p)\,\Gamma\, v(\bar{p}) \right|^2 = \mathrm{Tr}(\not{p}+m)\,\Gamma\,(\bar{\not{p}}-m')\,\bar{\Gamma}$$

$$\bar{\Gamma} \equiv \gamma_0 \Gamma^\dagger \gamma_0$$

$$[\bar{u}(p)\,\Gamma\, v(\bar{p})]^* = v^\dagger(\bar{p})\,\Gamma^\dagger \gamma_0^\dagger\, u(p)$$

$$= \bar{v}(\bar{p})\,\underbrace{\gamma_0 \Gamma^\dagger \gamma_0}_{\substack{\text{III} \\ \bar{\Gamma}}}\, u(p)$$

$$\sum \left| \bar{u}(p)\,\Gamma\, v(\bar{p}) \right|^2 = \sum \bar{u}_\alpha(p)\,\Gamma_{\alpha\beta}\, v_\beta(\bar{p})\, \bar{v}_\gamma(\bar{p})\, \bar{\Gamma}_{\gamma\delta}\, u_\delta(p)$$

$$\sum_{\text{spin}} u_\alpha(p)\,\bar{u}_\beta(p) = (\not{p}+m)_{\alpha\beta}$$

$$\sum v_\alpha(p)\,\bar{v}_\beta(p) = (\not{p}-m)_{\alpha\beta}$$

$$= (\not{p}+m)_{\delta\alpha}\,\Gamma_{\alpha\beta}\,(\bar{\not{p}}-\bar{m})_{\beta\gamma}\,\bar{\Gamma}_{\gamma\delta}$$

$$= \mathrm{Tr}(\not{p}+m)\,\Gamma\,(\bar{\not{p}}-\bar{m})\,\bar{\Gamma}$$

$\bar{\Gamma}$ vs Γ

$$\bar{\gamma}_\mu = \gamma_\mu$$

$$\overline{\gamma_\mu \gamma_\nu \cdots \gamma_\omega} = \gamma_\omega \cdots \gamma_\nu \gamma_\mu$$

$$\overline{\gamma_\mu \gamma_5} = \gamma_\mu \gamma_5$$

$$\bar{\gamma}_5 = -\gamma_5$$

Traceの計算の方法

★ $\Gamma = \{1, \gamma_\mu, \sigma_{\mu\nu}, \gamma_\mu\gamma_5, \gamma_5\}$ のうち 1以外は traceless.

1. $\mathrm{Tr}\,(\text{odd \# of } \gamma_\mu\text{'s}) = 0$

2. $\mathrm{Tr}\,1 = 4$

3. $\mathrm{Tr}\,\gamma_\mu\gamma_\nu = 4 g_{\mu\nu}$

4. $\mathrm{Tr}\,\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma = 4 S_{\mu\nu\rho\sigma}$

$$= 4\,(g_{\mu\nu}g_{\rho\sigma} + g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma})$$

または $\mathrm{Tr}\,\not{a}\not{b}\not{c}\not{d} = 4\,(a\cdot b\; c\cdot d + a\cdot d\; b\cdot c - a\cdot c\; b\cdot d)$

5. Tr (6個以上) は $\gamma_\mu\gamma_\nu = 2g_{\mu\nu} - \gamma_\nu\gamma_\mu$ を使って reduce

$$\mathrm{Tr}\,\not{a}_1\not{a}_2\cdots\not{a}_{2n} = \sum_{i=2}^{2n} (-1)^i\, a_1\cdot a_i \times \mathrm{Tr}\,\underbrace{\not{a}_2 \overset{\check{i}}{\cdots} \not{a}_{2n}}_{2n-2\text{個}}$$

6. $\mathrm{Tr}\,\gamma_5 = 0$

7. $\mathrm{Tr}\,\gamma_5\gamma_\mu\gamma_\nu = 0$

8. $\mathrm{Tr}\,\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma = 4i\,\epsilon_{\mu\nu\rho\sigma}$

Bj-D Convention

$$\begin{cases} \epsilon_{0123} = +1 \\ \epsilon^{0123} = -1 \end{cases}$$

諸々の公式

9. $\gamma_\lambda\gamma_\mu\gamma_\nu = S_{\lambda\mu\nu\rho}\gamma^\rho - i\,\epsilon_{\lambda\mu\nu\rho}\gamma^\rho\gamma_5$

10. $\gamma_\mu\gamma_\nu = g_{\mu\nu} - i\sigma_{\mu\nu}$

11. $\gamma_\alpha\gamma^\alpha = 4$

12. $\gamma_\alpha\gamma_\mu\gamma^\alpha = -2\gamma_\mu$

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{24}\,\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$$

13. $\gamma_\alpha \gamma_\mu \gamma_\nu \gamma^\alpha = 4 g_{\mu\nu}$

14. $\gamma_\alpha \gamma_\lambda \gamma_\mu \gamma_\nu \gamma^\alpha = -2 \gamma_\nu \gamma_\mu \gamma_\lambda$

おまけ

15. $\epsilon_{\mu\nu\rho\sigma} \epsilon^{\mu\nu\rho\sigma} = -24$

16. $\epsilon_{\mu\nu\rho\sigma} \epsilon^{\alpha\nu\rho\sigma} = -6\, g_\mu^\alpha$

17. $\epsilon_{\mu\nu\rho\sigma} \epsilon^{\alpha\beta\rho\sigma} = -2 (g_\mu^\alpha g_\nu^\beta - g_\mu^\beta g_\nu^\alpha)$

18. $\epsilon_{\mu\nu\rho\sigma} \epsilon^{\alpha\beta\gamma\sigma} = - g_\mu^\alpha g_\nu^\beta g_\rho^\gamma + g_\mu^\alpha g_\nu^\gamma g_\rho^\beta$ + 4 terms

19. $\epsilon_{\mu\nu\rho\sigma} \epsilon^{\alpha\beta\gamma\delta} = - g_\mu^\alpha g_\nu^\beta g_\rho^\gamma g_\sigma^\delta$ + 23 terms

20. $$g_{\alpha\lambda} \epsilon_{\mu\nu\rho\sigma} - g_{\alpha\mu} \epsilon_{\lambda\nu\rho\sigma} + g_{\alpha\nu} \epsilon_{\lambda\mu\rho\sigma}$$

$$- g_{\alpha\rho} \epsilon_{\lambda\mu\nu\sigma} + g_{\alpha\sigma} \epsilon_{\lambda\mu\nu\rho} = 0$$

(no 6^{th} rank totally antisym. tensor in 4 dim.)

$$g_{\alpha[\lambda} \epsilon_{\mu\nu\rho\sigma]} = 0$$

γ 行列の algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

$$[\gamma^\mu, \gamma^\nu] = -2i\sigma^{\mu\nu}$$

$$\{\gamma^\mu, \gamma_5\} = 0$$

$$[\sigma^{\mu\nu}, \gamma_5] = 0$$

$$\{\sigma^{\mu\nu}, \gamma_5\} = -i\epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma}$$ dual

$$(= 2\sigma^{\mu\nu}\gamma_5)$$

$$[\gamma^\lambda, \sigma^{\mu\nu}] = 2i(g^{\lambda\mu}\gamma^\nu - g^{\lambda\nu}\gamma^\mu)$$ γ^μ は Lorentz vector

$$\{\gamma^\lambda, \sigma^{\mu\nu}\} = 2\epsilon^{\lambda\mu\nu\rho}\gamma_\rho\gamma_5$$

$$[\sigma^{\mu\nu}, \sigma^{\rho\sigma}] = -2i(g^{\mu\rho}\sigma^{\nu\sigma} - g^{\mu\sigma}\sigma^{\nu\rho} - g^{\nu\rho}\sigma^{\mu\sigma} + g^{\nu\sigma}\sigma^{\mu\rho})$$

Lorentz 変換

$$\{\sigma^{\mu\nu}, \sigma^{\rho\sigma}\} = 2(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho} - i\epsilon^{\mu\nu\rho\sigma}\gamma_5)$$

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$

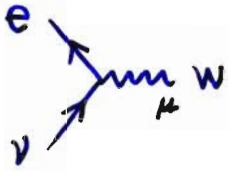
A54. W decay : $W \to e\bar{\nu}$

Interaction

$$\mathcal{L} = -\frac{g}{\sqrt{2}} \bar{e}_L \gamma_\mu \nu_L W^\mu$$

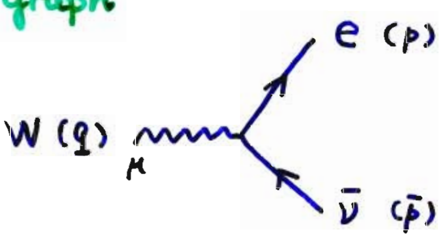
$$= -\frac{g}{2\sqrt{2}} \bar{e} \gamma_\mu (1-\gamma_5) \nu W^\mu$$

Feynman rule



$$-\frac{ig}{2\sqrt{2}} \gamma_\mu (1-\gamma_5)$$

Feynman graph



Amplitude

$$\mathcal{M} = (-i)\ \bar{u}(p) \frac{-ig}{2\sqrt{2}} \gamma_\mu (1-\gamma_5) v(\bar{p})\ \epsilon^\mu$$

$$= -\frac{g}{2\sqrt{2}} \bar{u}(p) \gamma_\mu (1-\gamma_5) v(\bar{p}) \epsilon^\mu$$

Decay width

$$\Gamma = \frac{1}{2m_W} \ \underset{\substack{\uparrow \\ 2J+1=3 \\ \text{for } W}}{\frac{1}{3}} \sum_{\text{spins}} |\mathcal{M}|^2 \ d\Phi_2$$

$$d\Phi_2 = \frac{1}{32\pi^2} d\Omega \quad (m_e \to 0 \text{ taken})$$

$$= \frac{1}{8\pi}$$

$$= \frac{1}{48\pi m_W} \sum_{\text{spins}} |\mathcal{M}|^2$$

$$\sum |\mathcal{M}|^2 = \frac{g^2}{8} \sum \left| \bar{u}(p) \gamma_\mu (1-\gamma_5) v(\bar{p}) \epsilon^\mu \right|^2$$

$$= \frac{g^2}{8} \mathrm{Tr}\, \not{p} \gamma_\mu (1-\gamma_5) \bar{\not{p}} \gamma_\nu (1-\gamma_5) \underbrace{\sum_{\mathrm{spin}} \epsilon^\mu \epsilon^{*\nu}}_{-g^{\mu\nu} + \frac{q^\mu q^\nu}{m_W^2}}$$

$$= \frac{2g^2}{8} \mathrm{Tr}\, \not{p} \gamma_\mu \bar{\not{p}} \gamma_\nu (1-\gamma_5) \left[-g^{\mu\nu} + \frac{q^\mu q^\nu}{m_W^2} \right]$$

- γ_5 の部分の trace は $\mu \leftrightarrow \nu$ 反対称のため消える.
- $\mathrm{Tr}\, \not{p} \not{q} \bar{\not{p}} \not{q} = \mathrm{Tr}\, \not{p}(\underbrace{\not{p}}_{p^2=0} + \underbrace{\bar{\not{p}})\bar{\not{p}}}_{\bar{p}^2=0} \not{q} = 0$ なので $q^\mu q^\nu$ は効かない

$$= -\frac{g^2}{4} \mathrm{Tr}\, \not{p} \underbrace{\gamma_\mu \bar{\not{p}} \gamma^\mu}_{-2\bar{\not{p}}}$$

$$= \frac{g^2}{2} \underbrace{\mathrm{Tr}\, \not{p} \bar{\not{p}}}_{4 p\cdot\bar{p} \,=\, 2q^2 \,=\, 2m_W^2}$$

$$= g^2 m_W^2$$

Width : $\Gamma = \dfrac{1}{48\pi m_W} \cdot g^2 m_W^2 = \dfrac{g^2 m_W}{48\pi} = \dfrac{\alpha m_W}{12 \sin^2\theta_W}$

$e = g \sin\theta_W \qquad \alpha = \dfrac{e^2}{4\pi}$

問題：μ decay

$$\mu \to e\bar{\nu}_e\nu_\mu$$

- $\mu\bar{\nu}_\mu W$ の相互作用は $e\bar{\nu}_e W$ と同じ

1. $\mu \to e\bar{\nu}_e\nu_\mu$ の amplitude を書け.

2. $m_\mu^2 \ll m_W^2$ の近似（→Fermi 相互作用），$m_e = m_{\nu_e} = m_{\nu_\mu} = 0$ として decay rate を計算せよ。

   $\left(d\Phi_3 = \dfrac{m_\mu^2}{128\pi^3} dz_1 dz_2 \text{ を使って} \right)$

   答： $\Gamma = \dfrac{G_F^2 m_\mu^5}{192\pi^3}$

   但し $\dfrac{G_F}{\sqrt{2}} = \dfrac{g^2}{8m_W^2}$

3. $m_e \neq 0$ のときの decay rate を求めよ.

4. $m_\mu - m_e \to +0$ のとき、decay rate は Q value ($m_\mu - m_e$) の何乗で 0 になるか。それはなぜか。

Parton Model

Quark/gluon cross section ← Perturbation theory

↓ Parton model

Hadron cross section

Calculate $\sigma(h_1 + h_2 \to X + \text{anything})$

from $\sigma(i + j \to X)$ $\quad i, j$: quark/gluon

$$\boxed{d\sigma(h_1+h_2\to X+\text{any}) = \int dx_1 dx_2\, f_{i/h_1}(x_1)\, f_{j/h_2}(x_2)\, d\sigma(i+j\to X)\Big|_{\hat{s}=x_1x_2s}}$$

measured mostly from deep inelastic scattering $\ell N \to \ell X$

$f_{i/h}(x)$: structure function or parton distribution

i ← parton, h ← hadron

x : momentum fraction $\quad 0<x<1$

$k_i = x\, k_h$ (mass neglected)

momentum sum rule $\quad \sum_i \int_0^1 dx\, f_{i/h}(x) = 1$

s : E_{cm}^2 for hadron system

$\hat{s}$: E_{cm}^2 for parton subprocess $i+j\to X$

^ is often used for subprocess quantities

Parton Luminosity

$$\frac{d\mathcal{L}}{d\tau} = \int_0^1 dx_1 \int_0^1 dx_2\, f_1(x_1) f_2(x_2)\, \delta(x_1 x_2 - \tau)$$

$\tau = x_1 x_2$ $\quad\quad$ $\tau = \hat{s}/s$

$y = \frac{1}{2}\ln\frac{x_1}{x_2}$ $\quad\quad$ rapidity

$$dx_1 dx_2 = d\tau\, dy$$

$$= \int_{-y_{max}}^{y_{max}} dy\, f(\sqrt{\tau}e^{y})\, f(\sqrt{\tau}e^{-y})$$

$$y_{max} = \frac{1}{2}\ln\frac{1}{\tau}$$

$\tau \frac{d\mathcal{L}}{d\tau}$ gives the effective parton "luminosity" in one collision

for a subprocess with $\sim \hat{s} = \tau s$

$$\boxed{\hat{s}\frac{d\sigma}{d\hat{s}} = \tau\frac{d\mathcal{L}}{d\tau}\cdot\hat{\sigma}\Big|_{\hat{s}=\tau s}}$$

$\frac{d\sigma}{d\ln\hat{s}}$

例. $p\bar{p} \to W^- + \text{anything}$

簡単のため proton 中の d quark と antiproton 中の $\bar{u}$ quark から W^- ができる場合を考える (sea quark, s quark, ..., higher order QCD 無視)

Subprocess $d + \bar{u} \to W^-$

$$\hat{\sigma} = \frac{\pi^2 \alpha}{3 \sin^2\theta_W} \delta(\hat{s} - m_W^2) \qquad (\text{既出})$$

Use

$$d\sigma = \int dx_1 dx_2 \, d_p(x_1) \, u_p(x_2) \, d\hat{\sigma}(\hat{s} = x_1 x_2 s)$$

$d_p(x) \equiv f_{d/p}(x)$

$\bar{u}_{\bar{p}}(x) = u_p(x)$ (charge conjugation)

$$= \int_0^1 d\tau \int dy \, d(x_1) \, u(x_2) \, \frac{\pi^2 \alpha}{3 \sin^2\theta_W} \delta(\tau s - m_W^2)$$

$$= \frac{\pi^2 \alpha}{3 s \sin^2\theta_W} \int dy \, d(x_1) \, u(x_2)$$

$$= \frac{\pi^2 \alpha}{3 m_W^2 \sin^2\theta_W} \cdot \tau \frac{d\mathcal{L}^{d\bar{u}}}{d\tau} \bigg|_{\tau = \frac{m_W^2}{s}}$$

$$= \frac{\sqrt{2} \pi G_F}{3} \cdot \tau \frac{d\mathcal{L}}{d\tau}$$