3

THE STANDARD MODEL

素粒子の相互作用 : Now and then

Pre historic

Strong

核力

湯川 1935. ....

中間子論

$m_\pi \sim 10^{-1}$ GeV

$g_{NN\pi} \gg 1$

Electromagnetic

Maxwell

QED

$\alpha = \frac{1}{137}$

Weak

β decay

Fermi 1934

not renormalizable

$G_F \sim 10^{-5}\,\mathrm{GeV}^{-2}$

Gravitation

Newton → Einstein 1915

素粒子の相互作用 : Now and then

	Pre historic	Modern
Strong 核力	湯川 1935 . .... 中間子論 ← bound state problem $m_\pi \sim 10^{-1}$ GeV $g_{NN\pi} \gg 1$	QCD gluons $\Lambda \sim 10^{-1}$ GeV
Electromagnetic Maxwell	QED $\alpha = \frac{1}{137}$	GWS $SU(2) \times U(1)$ γ
Weak β decay	Fermi 1934 → not renormalizable effective theory $G_F \sim 10^{-5}\,\mathrm{GeV}^{-2} \sim \frac{1}{m_W^2}$ $m_W \sim 10^2$ GeV	Charged current $W^{\pm}$ Neutral current 1973 Z
Gravitation Newton → Einstein 1915		

素粒子の相互作用 : Now and then

Pre historic — Modern — $\boxed{\text{Standard Model}}$

Strong
核力
Confining

湯川 1935 . . . .
中間子論 ← bound state problem
$m_\pi \sim 10^{-1}$ GeV
$g_{NN\pi} \gg 1$

QCD
gluons
$\Lambda \sim 10^{-1}$ GeV

Electromagnetic
Maxwell
Coulomb
(massless photon)

QED
$\alpha = \frac{1}{137}$

GWS
SU(2) × U(1)
γ

Weak
β decay
Spontaneously broken

Fermi 1934 → Charged current $W^{\pm}$
not renormalizable
↗ effective theory
$G_F \sim 10^{-5}\,\mathrm{GeV}^{-2} \sim \frac{1}{m_W^2}$
$m_W \sim 10^2$ GeV

Neutral current 1973 Z

Gauge interactions
but 3 couplings
(→ GUT ?)

Yukawa interactions
Fermion masses
Flavor, CP violation .

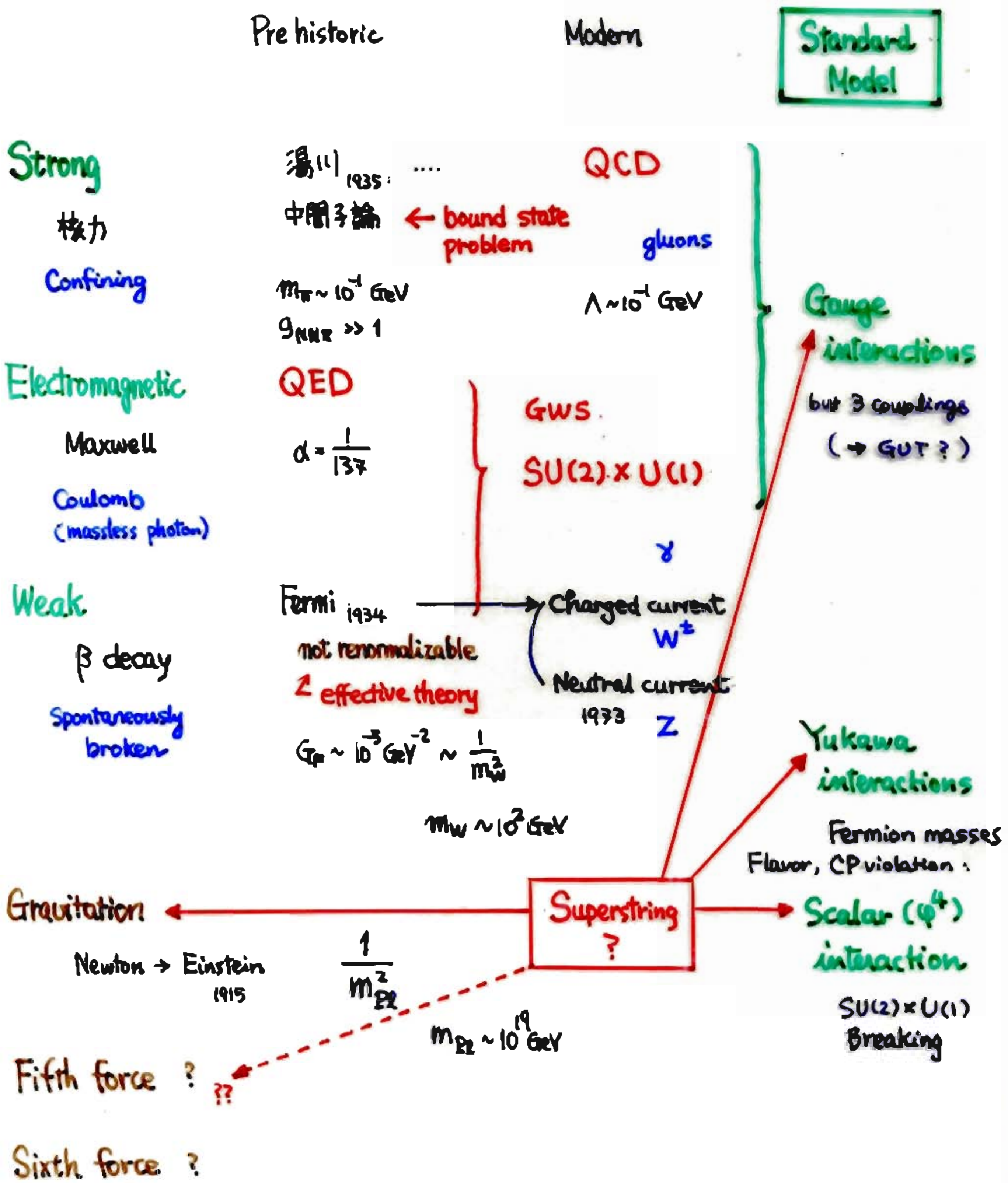
Scalar (φ^4) interaction
SU(2) × U(1) Breaking

Gravitation
Newton → Einstein 1915
$\frac{1}{m_{Pl}^2}$
$m_{Pl} \sim 10^{19}$ GeV

Fifth force ?

Sixth force ?

素粒子の相互作用 : Now and then and future ?



QCD (Quantum Chromodynamics)

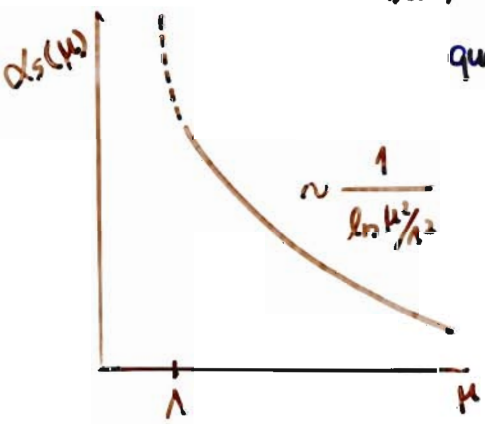
Hadrons … "bound states" of quarks and gluons

- Asymptotic freedom (effective gauge coupling → 0 at high energy)

  ⇨ Interactions of quarks, gluons "visible" at high energies

  Perturbation theory applicable

  But, usually need parameters connecting quark/gluon ↔ hadron (should be measured)



Structure functions (parton distributions)
decay constant (f_π, …
wave function (quarkonium)
matrix elements
⋮

exception : $\sigma(e^+e^- \to \text{hadrons})$

- Infrared "slavery"

  → Color confinement

Important : Separation of short- and long-distance physics ("Factorization")

short-	long-distance
perturbative quark/gluon	non-perturbative hadron
calculable	not yet calculable (lattice …)

QCD Lagrangian

Gauge group : $SU(3)$

gauge boson (gluon) 8

fermions (quarks) $(3_L + 3^*_L) \times n_f$

$\simeq (3_L + 3_R) \times n_f$

n_f : # flavors

Parameters

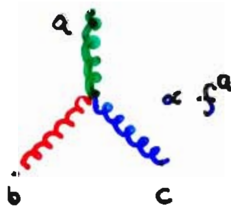
gauge coupling g_s or $\alpha_s = \dfrac{g_s^2}{4\pi}$ or Λ_{QCD}

quark masses $m_u, m_d, m_s, m_c, m_b, m_t$

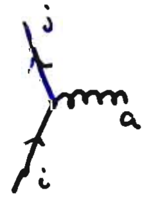
(generated by "weak interactions" or Yukawa interactions)

$$\mathcal{L} = -\frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} + \sum_{\text{flavor}} \bar{q}(i\not{D} - m_q) q \quad (+ \text{ gauge fixing } + \text{ ghost})$$

$\Downarrow$ $\Downarrow$



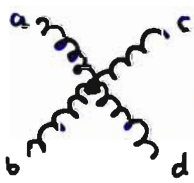
$\propto f^{abc}$



$\propto T^a_{ij}$

$T^a_{ij} = \frac{1}{2}\lambda^a_{ij}$

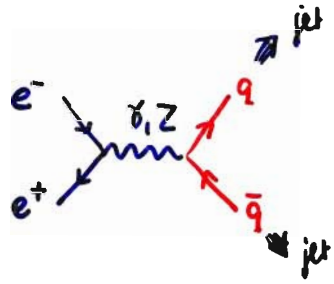
($SU(3)$ Gell-Mann matrix)



$\propto f^{abe} f^{cde}$

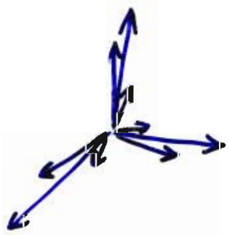
Visualizing quarks and gluons

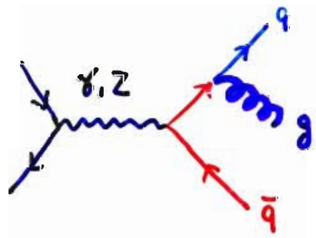
$e^+e^- \to 2$ jets



inferred from sphericity distribution at SPEAR ($\sqrt{s} \lesssim 7$ GeV)
clearly visible at PETRA ($\sqrt{s} \sim 30-40$ GeV)

$e^+e^- \to 3$ jets

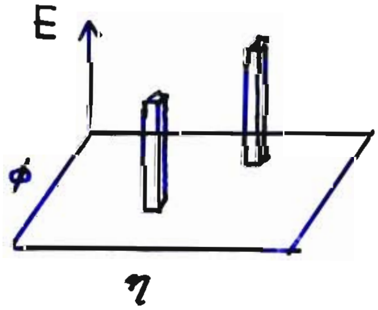


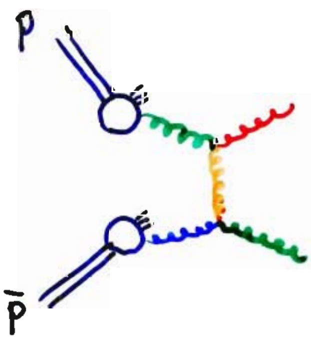


first seen at PETRA

3 jet rate $\sim \alpha_s \times$ (2 jet rate) $\qquad q\bar{q}g$ coupling

$p\bar{p} \to 2$ jet + anything





seen at CERN $p\bar{p}$ collider

indicate gluon self coupling ?

$SU(2)_L \times U(1)_Y \to U(1)_Q$: Gauge-Higgs system

(minimal)

Fermions play no role ($\leftrightarrow$ Technicolor)

Start from a Higgs field φ : SU(2) doublet, nonzero Y charge (The simplest Higgs structure)

$$\varphi = \frac{1}{\sqrt{2}} \begin{pmatrix} i(\varphi_1 - i\varphi_2) \\ \varphi_0 - i\varphi_3 \end{pmatrix}$$

SU(2) 変換 $\quad \varphi \to U\varphi \qquad U = \exp(i\tau_a \theta_a/2)$ $\qquad \tau_a$: Pauli 行列

θ : パラメータ

U(1) $\quad \varphi \to e^{i\theta}\varphi$

Potential : should be $SU(2)\times U(1)$ invariant

$$\underline{\mathcal{L}_{\text{Higgs}} = (D_\mu\varphi)^\dagger D_\mu\varphi - V(\varphi)}$$

quadratic term $\quad 2\times 2 = 1 + 3 \qquad 2^* \sim 2$

$\varphi^\dagger\varphi$: SU(2) singlet, zero U(1) charge

$\varphi^T i\tau_2\varphi$ (+h.c.) $= 0$ $\quad$ though SU(2) singlet
Bose statistics (2 is pseudoreal)
(nonzero hypercharge anyway)

cubic $\quad 2\times 2\times 2 \not\ni 1$

quartic $\quad 2^4 = (1+3)\times(1+3) = 1 + 3 + 3 + (1+3+5)$

$(\varphi^\dagger\varphi)^2$: OK

$(\varphi^\dagger\tau^a\varphi)^2 \propto (\varphi^\dagger\varphi)^2 \qquad \because (\tau^a)_{ij}(\tau^a)_{kl} = 2\delta_{il}\delta_{kj} - \delta_{ij}\delta_{kl}$

$|\varphi^T i\tau_2\varphi|^2 = 0$

$|\varphi^T i\tau_2\tau^a\varphi|^2 \propto (\varphi^\dagger\varphi)^2$

The most general renormalizable potential is thus

$$\underline{V(\varphi) = \mu^2\varphi^\dagger\varphi + \lambda(\varphi^\dagger\varphi)^2}.$$

Symmetry of the Higgs potential

$V(\varphi) = \mu^2 \varphi^\dagger\varphi + \lambda(\varphi^\dagger\varphi)^2$ is a function of $\varphi^\dagger\varphi$

$\varphi^\dagger\varphi = \frac{1}{2}(\varphi_0^2+\varphi_1^2+\varphi_2^2+\varphi_3^2) \rightarrow$ O(4) symmetric

$O(4) \sim O(3)\times O(3) \sim SU(2)\times SU(2) \supset SU(2)_L \times U(1)_Y$

how?

Matrix notation

$$\Phi \equiv \frac{1}{\sqrt{2}}(\varphi_0 + i\tau_a\varphi_a) \qquad a=1,2,3 \quad (\varphi_i : \text{real})$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} \varphi_0+i\varphi_3 & i(\varphi_1-i\varphi_2) \\ i(\varphi_1+i\varphi_2) & \varphi_0-i\varphi_3 \end{pmatrix}$$

見かけ上 complex 4成分, but 実質は real 4成分 → Reality constraint

$$\boxed{\tau_2\Phi^*\tau_2 = \Phi}$$

$\because)\ \tau_2\Phi^*\tau_2 = \frac{1}{\sqrt{2}}(\varphi_0 - i\tau_2\tau_a^*\tau_2\varphi_a)$

$\tau_2\tau_a^*\tau_2 = -\tau_a$

$= \frac{1}{\sqrt{2}}(\varphi_0 + i\tau_a\varphi_a)$

$= \Phi$

2成分記法との関係

$$\varphi = \frac{1}{\sqrt{2}}\begin{pmatrix} i(\varphi_1-i\varphi_2) \\ \varphi_0-i\varphi_3 \end{pmatrix}$$

$$\tilde{\varphi} \equiv i\tau_2\varphi^* = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} -i(\varphi_1+i\varphi_2) \\ \varphi_0+i\varphi_3 \end{pmatrix} = \begin{pmatrix} \varphi_0+i\varphi_3 \\ i(\varphi_1+i\varphi_2) \end{pmatrix}$$

$$\boxed{\Phi = (\tilde{\varphi}\ \ \varphi)}$$

$\tilde{\varphi}$: 形を整えた φ^*

$\Phi = \frac{1}{\sqrt{2}}(\varphi_0\mathbb{1} + i\tau_a\varphi_a) \qquad \Phi^\dagger = \frac{1}{\sqrt{2}}(\varphi_0\mathbb{1} - i\tau_b\varphi_b)$

$\Phi^\dagger\Phi = \frac{1}{2}\left[\varphi_0^2\mathbb{1} + \tau_a\tau_b\varphi_a\varphi_b\right] \qquad \tau_a\tau_b = \delta_{ab} + i\epsilon_{abc}\tau_c$

$= \frac{1}{2}\left(\sum_{i=0}^{3}\varphi_i^2\right)\mathbb{1} \qquad \Phi\Phi^\dagger = \Phi^\dagger\Phi$

SU(2) × U(1) 変換

$SU(2)_L \quad \varphi \to U\varphi \qquad \tilde{\varphi} \to U\tilde{\varphi}$

$U(1)_Y \quad \varphi \to e^{i\theta}\varphi \qquad \tilde{\varphi} \to e^{-i\theta}\tilde{\varphi}$

$\because)\quad U = \exp(i\tau_a\theta^a)$

$$\varphi^* \to U^*\varphi^* = \exp(-i\tau_a^*\theta^a)\,\varphi^*$$

$$\tilde{\varphi} \to i\tau_2 \exp(-)\,\varphi^* = \exp(-i\tau_2\tau_a^*\tau_2\theta^a)\,i\tau_2\varphi^* = \exp(+i\tau_a\theta^a)\,\tilde{\varphi} = U\tilde{\varphi}$$

$$\boxed{\begin{array}{l} \Phi \to U\Phi V^\dagger \\ U \in SU(2) \\ V = \begin{pmatrix} e^{i\alpha} & \\ & e^{-i\alpha} \end{pmatrix} = e^{i\alpha\tau_3} \in U(1) \end{array}}$$

Invariants

quadratic

$$\underline{\mathrm{Tr}\,\Phi^\dagger\Phi} \to \mathrm{Tr}\,V\Phi^\dagger U^\dagger U\Phi V^\dagger = \mathrm{Tr}\,\Phi^\dagger\Phi \quad : SU(2)\times U(1)\text{-invariant}$$

$$\mathrm{Tr}\,\Phi^\dagger\Phi = \frac{1}{2}\sum\varphi_i^2\,\mathrm{Tr}\,\mathbf{1} = \sum\varphi_i^2 = 2\varphi^\dagger\varphi$$

quartic

$$\underline{(\mathrm{Tr}\,\Phi^\dagger\Phi)^2} = (\sum\varphi_i^2)^2$$

$$\mathrm{Tr}\,\Phi^\dagger\Phi\Phi^\dagger\Phi = \frac{1}{4}(\sum\varphi_i^2)^2\,\mathrm{Tr}\,\mathbf{1} = \frac{1}{2}(\mathrm{Tr}\,\Phi^\dagger\Phi)^2$$ 独立でない

その他

$$\det\Phi \to \det U\Phi V^\dagger = \det U\,\det\Phi\,\det V^\dagger = \det\Phi$$

$$\det\Phi = \frac{1}{2}\sum\varphi_i^2 = \frac{1}{2}\mathrm{Tr}\,\Phi^\dagger\Phi$$

$$(\mathrm{Tr}\,\Phi^\dagger\tau_a\Phi)^2 = 0$$

$$\mathrm{Tr}\,\Phi^\dagger\tau_a\Phi\Phi^\dagger\tau_a\Phi = \frac{3}{2}(\sum\varphi_i^2)^2$$

etc.

Higgs potential

$$V(\varphi) = \mu^2 \varphi^\dagger\varphi + \lambda(\varphi^\dagger\varphi)^2$$

$$= \frac{1}{2}\mu^2 \, \mathrm{Tr}\,\Phi^\dagger\Phi + \frac{1}{4}\lambda\left(\mathrm{Tr}\,\Phi^\dagger\Phi\right)^2$$

has larger (global) symmetry than $SU(2)\times U(1)$: $SU(2)_L \times SU(2)_R \sim O(4)$

$$\Phi \to U\Phi V^\dagger \qquad U, V \in SU(2)$$

$$\mathrm{Tr}\,\Phi^\dagger\Phi \to \mathrm{Tr}\,V\Phi^\dagger U^\dagger U \Phi V^\dagger = \mathrm{Tr}\,\Phi^\dagger\Phi$$

Spontaneous Breaking of $SU(2)\times U(1)$

choose $\mu^2 < 0$

$\lambda > 0$ (stability constraint : if not, potential unbounded from below)

V は $\mathrm{Tr}\,\Phi^\dagger\Phi$ のみの関数

$$\frac{\partial V}{\partial\,\mathrm{Tr}\,\Phi^\dagger\Phi} = 0 = \frac{1}{2}\mu^2 + \frac{1}{2}\lambda\,\mathrm{Tr}\,\Phi^\dagger\Phi \quad : \text{extremum condition}$$

解 : $\mathrm{Tr}\,\Phi^\dagger\Phi = \dfrac{-\mu^2}{\lambda}$

or $\displaystyle\sum_{i=0}^{3} \varphi_i^2 = \frac{-\mu^2}{\lambda} \equiv v^2$

一般性を失うことなく $\varphi_0 = v$, $\varphi_a = 0$ $(a=1,2,3)$ ととれる

真空期待値 : $\boxed{\langle\Phi\rangle = \dfrac{v}{\sqrt{2}}\,\mathbb{1}}$

Unbroken (gauge) symmetry

$\langle\Phi\rangle$ を動かさない(gauge)変換

$$\langle\Phi\rangle = \frac{v}{\sqrt{2}}\mathbb{1} \rightarrow U\langle\Phi\rangle V^\dagger = \frac{v}{\sqrt{2}}UV^\dagger$$

$= \frac{v}{\sqrt{2}}\mathbb{1}$ であるためには $UV^\dagger = 1$

$$U = V = e^{i\tau_3\alpha} \in U(1)$$

gauge: $SU(2)_L \times U(1)_Y \rightarrow U(1)$

Global unbroken symmetry

$$U = V \in SU(2)$$

Sometimes called "custodial SU(2)"

global: $SU(2)_L \times SU(2)_R \rightarrow SU(2)_D$: diagonal SU(2) group

注意: This group is the symmetry of the Higgs potential only.

In the full theory, $SU(2)_R$ and $SU(2)_D$ is explicitly broken by the $U(1)_Y$ gauge interactions ($U(1)_Y$ distinguishes the 3rd direction)

However, there are situations in which this group appears as an approximate symmetry:

- Large m_H limit ($\lambda \gg g^2$)
- $\rho = 1$ is related to the symmetry

Higgs mass matrix

Higgs potential

$$V = \mu^2 \varphi^\dagger \varphi + \lambda (\varphi^\dagger \varphi)^2$$

$$= \frac{1}{2}\mu^2 \sum_{k=0}^{3} \varphi_k^2 + \frac{1}{4}\lambda \left(\sum_k \varphi_k^2\right)^2 \qquad \varphi_k : \text{real}$$

Extremum

$$\frac{\partial V}{\partial \varphi_i} = 0 \quad \Rightarrow \quad \varphi_i \underline{\left(\mu^2 + \lambda \sum_k \varphi_k^2\right)} = 0 \qquad \text{for all } i$$

assume $\mu^2 < 0$

$$\sum_k \varphi_k^2 = \frac{-\mu^2}{\lambda} \equiv v^2 \quad : \text{minimum (vacuum)}$$

$(\varphi_i = 0 \ : \ \text{maximum})$

Mass matrix

$$V = \cdots + \frac{1}{2} M_{ij}^2 \varphi_i \varphi_j + \cdots$$

$$M_{ij}^2 = \left.\frac{\partial^2 V}{\partial \varphi_i \partial \varphi_j}\right|_{\varphi = \langle\varphi\rangle} = \left.\delta_{ij} \underbrace{\left(\mu^2 + \lambda \sum \varphi_k^2\right)}_{0} + 2\lambda \varphi_i \varphi_j \right|_{\varphi = \langle\varphi\rangle}$$

Vacuum : $\langle \varphi_0 \rangle = v$, $\langle \varphi_a \rangle = 0$ $(a = 1,2,3)$ ととって一般性を失わない

$$M_{ij}^2 = 2\lambda v^2 \delta_{i0}\delta_{j0} = -2\mu^2 \delta_{i0}\delta_{j0} = \begin{pmatrix} -2\mu^2 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

$H \equiv \varphi_0 - v$ として $m_H^2 = -2\mu^2 > 0$

$m_{\varphi_a}^2 = 0$ ← massless states (Goldstone boson)

$\#$ Goldstone boson $= 3 = \dim(G/H)$

full group ↲ ↳ unbroken subgroup

$$G = SU(2)_L \times SU(2)_R \rightarrow H = SU(2)_D$$

$$SU(2)_L \times U(1)_Y \rightarrow U(1)_Q \quad (\text{gauged version})$$

Goldstone's theorem

Spontaneous symmetry breaking $\Rightarrow$ massless bosons
$G \rightarrow H$ (Goldstone bosons)

Goldstone bosons = dim G/H

Proof (classical version)

$\langle\phi\rangle$: a vacuum $\quad \left.\dfrac{\partial V}{\partial \phi_i}\right|_{\phi=\langle\phi\rangle} = 0$

$g \in G \qquad g\langle\phi\rangle = \langle\phi\rangle'$

Potential V (or Lagrangian) は g-invariant

$\Rightarrow$ $\langle\phi\rangle'$ も (another) vacuum

Unbroken subgroup H

$h \in H \Rightarrow h\langle\phi\rangle = \langle\phi\rangle$ ($\langle\phi\rangle$ を動かさない)

$\rightarrow$ Vacuum states のつくる manifold は G/H

$\rightarrow$ G/H 上では V の値は const. (energy density of vacuum)

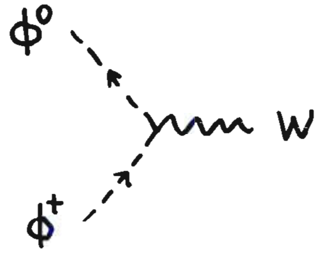
$\rightarrow$ $\langle\phi\rangle$ における G/H 方向の excitation は massless

$\rightarrow$ # massless states = dim G/H

Higgs mechanism

If the broken symmetry is gauged,
the Goldstone boson disappear
and the corresponding gauge boson becomes massive.

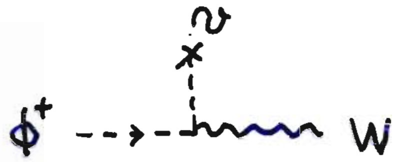
Higgs-gauge interaction $\quad D_\mu \varphi^\dagger D_\mu \varphi \qquad \varphi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$



Sym. breaking

$\longrightarrow$

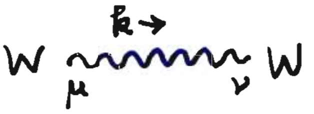
$\phi^0 = \frac{1}{\sqrt{2}}(v + H + i\chi)$



$$-\frac{ig}{\sqrt{2}}\left(\phi^{0*} \overleftrightarrow{\partial_\mu} \phi^+\right) W_\mu \qquad\qquad -\frac{i}{2} g v \left(\partial_\mu \phi^+\right) W_\mu$$

"mixing" of $\partial_\mu \phi^+$ and W_μ^+

Gauge boson propagator



$$= \frac{-i}{k^2}\left(\underbrace{g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}}_{\text{"transverse"}} + \alpha \underbrace{\frac{k_\mu k_\nu}{k^2}}_{\text{"longitudinal"}} \right)$$

(4-dim sense)

vacuum polarization

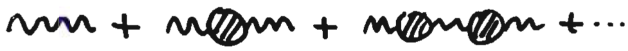
$$= -i\left(g_{\mu\nu} k^2 - k_\mu k_\nu\right) \Pi(k^2)$$

↑ from gauge invariance

$$= \frac{-i}{k^2}\left(g_{\mu\rho} - \frac{k_\mu k_\rho}{k^2} + \alpha \frac{k_\mu k_\rho}{k^2}\right)(-i)\left(g_{\rho\sigma} k^2 - k_\rho k_\sigma\right)\Pi(k^2)\, \frac{-i}{k^2}\left(g_{\sigma\nu} - \cdots\right.$$

$$= \frac{+i}{k^2}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\Pi(k^2)$$

no longitudinal part due to gauge invariance

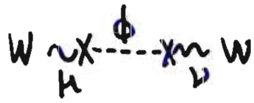


$$= \frac{-i}{k^2(1+\Pi(k^2))}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) + \frac{-i}{k^2}\,\alpha\,\frac{k_\mu k_\nu}{k^2}$$

↳ gauge boson mass = 0 (as far as $\Pi(0)$ is regular)

↳ longitudinal part は radiative correction をうけない

Goldstone contribution to vacuum polarization



$$= \frac{-i}{2} g v k_\mu \cdot \frac{i}{k^2} \cdot \frac{-i}{2} g v k_\nu = -\frac{i}{4} g^2 v^2 \frac{k_\mu k_\nu}{k^2}$$

$$\updownarrow$$

$$-i\,(g_{\mu\nu}k^2 - k_\mu k_\nu)\,\Pi(k^2)$$

↓ non-pole term

$$\Pi(k^2) \underset{k^2\sim 0}{\sim} -\frac{1}{4} g^2 v^2 \frac{1}{k^2}$$

$$\frac{1}{k^2(1+\Pi(k^2))} = \frac{1}{k^2\left(1 - \frac{1}{4}g^2v^2\frac{1}{k^2}\right)} = \frac{1}{k^2 - \underbrace{\tfrac{1}{4}g^2v^2}_{\text{W mass}}}$$

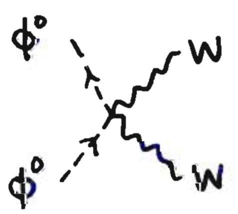
gauge boson mass shifted from 0

because of the Goldstone pole

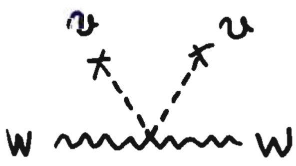
$$m_W^2 = \frac{1}{4} g^2 v^2$$

In practice, it is convenient to cancel the $\partial_\mu \phi W_\mu$ term in the Lagrangian by a gauge fixing term (R_ξ gauge)

The gauge boson mass term is contained in $D_\mu \varphi^\dagger D_\mu \varphi$



$$\xrightarrow{\langle \phi^0 \rangle = \frac{1}{\sqrt{2}} v}$$



$$\frac{1}{2} g^2 \phi^{0*} \phi^0 W_\mu^\dagger W_\mu \qquad\qquad \frac{1}{4} g^2 v^2 W_\mu^\dagger W_\mu$$

Gauge boson mass

$$D_\mu \varphi = \left(\partial_\mu + i \underset{\substack{\uparrow \\ \text{SU(2) coupling}}}{g} \frac{\tau_a}{2} W_\mu^a + i \underset{\substack{\uparrow \\ \text{U(1) coupling}}}{g'} \frac{1}{2} B_\mu \right) \varphi$$

There is no inherent normalization of U(1) (unlike simple groups)

→ we can choose the $U(1)_Y$ hypercharge of φ at will

by redefining g' (We take $\frac{1}{2}$ as convention) !

Matrix notation

$$D_\mu \Phi = \partial_\mu \Phi + \frac{i}{2} g \, \tau_a W_\mu^a \Phi - \frac{i}{2} g' \, \Phi \tau_3 B_\mu$$

$$D_\mu \Phi^\dagger = \partial_\mu \Phi^\dagger - \frac{i}{2} g \, \Phi^\dagger \tau_a W_\mu^a + \frac{i}{2} g' \, \tau_3 B_\mu \Phi^\dagger$$

Gauge boson mass terms come from the Higgs kinetic term

$$\mathcal{L} = \frac{1}{2} \mathrm{Tr} \, D_\mu \langle \Phi \rangle^\dagger D_\mu \langle \Phi \rangle \qquad \langle \Phi \rangle = \frac{v}{\sqrt{2}} \mathbf{1}$$

$$D_\mu \langle \Phi \rangle = \frac{i}{2} \frac{v}{\sqrt{2}} \left(g \, \tau_a W_\mu^a - g' \tau_3 B_\mu \right)$$

$$= \frac{i v}{2\sqrt{2}} \tau_a X_\mu^a \qquad \begin{cases} X_\mu^{1,2} = g W_\mu^{1,2} \\ X_\mu^3 = g W_\mu^3 - g' B_\mu \end{cases}$$

$$= \frac{1}{16} v^2 \, \mathrm{Tr} (\tau_a X_\mu^a)^2$$

$$= \frac{1}{8} v^2 (X_\mu^a)^2$$

$$= \frac{1}{8} v^2 \left\{ g^2 \left[(W_\mu^1)^2 + (W_\mu^2)^2 \right] + (g W_\mu^3 - g' B_\mu)^2 \right\}$$

Note: もし full $SU(2)_L \times SU(2)_R$ が gauge 化されていれば

$$\frac{1}{8} v^2 \sum_{a=1}^{3} (g W_\mu^a - g' B_\mu^a)^2 \quad : \quad SU(2)_D \text{ invariant}$$

define

$$W_\mu = \frac{1}{\sqrt{2}}(W_\mu^1 + i W_\mu^2)$$

$$Z_\mu = \frac{1}{\sqrt{g^2+g'^2}}(g W_\mu^3 - g' B_\mu)$$

$$\mathcal{L} = \frac{1}{4} g^2 v^2 W_\mu^\dagger W_\mu + \frac{1}{8}(g^2+g'^2) v^2 Z_\mu Z_\mu \quad = m_W^2 W_\mu^\dagger W_\mu + \frac{1}{2} m_Z^2 Z_\mu Z_\mu$$

$$m_W = \frac{1}{2} g v$$

$$m_Z = \frac{1}{2}\sqrt{g^2+g'^2}\, v \quad > m_W$$

直交する state

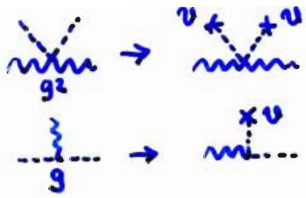
$$A_\mu = \frac{1}{\sqrt{g^2+g'^2}}(g' W_\mu^3 + g B_\mu)$$

は massless = photon

$$\begin{cases} Z_\mu = W_\mu^3 \cos\theta_W - B_\mu \sin\theta_W \\ A_\mu = W_\mu^3 \sin\theta_W + B_\mu \cos\theta_W \end{cases} \qquad \tan\theta_W = \frac{g'}{g}$$

$$\begin{cases} W_\mu^3 = Z_\mu \cos\theta_W + A_\mu \sin\theta_W \\ B_\mu = -Z_\mu \sin\theta_W + A_\mu \cos\theta_W \end{cases}$$

4 Higgs fields φ_i $(i=0,\cdots,3)$ のうち 3つ $(\varphi_1, \varphi_2, \varphi_3)$ は gauge bosons $(W^\pm, Z)$ の「たて波成分」となり、$\varphi_0 = H$ だけが physical な粒子として残る.



= gauge boson mass $g^2 v^2 A_\mu^2$

= gauge boson -scalar mixing $g v A_\mu \partial_\mu \phi$

(gauge fixing で消去できる)

unbroken

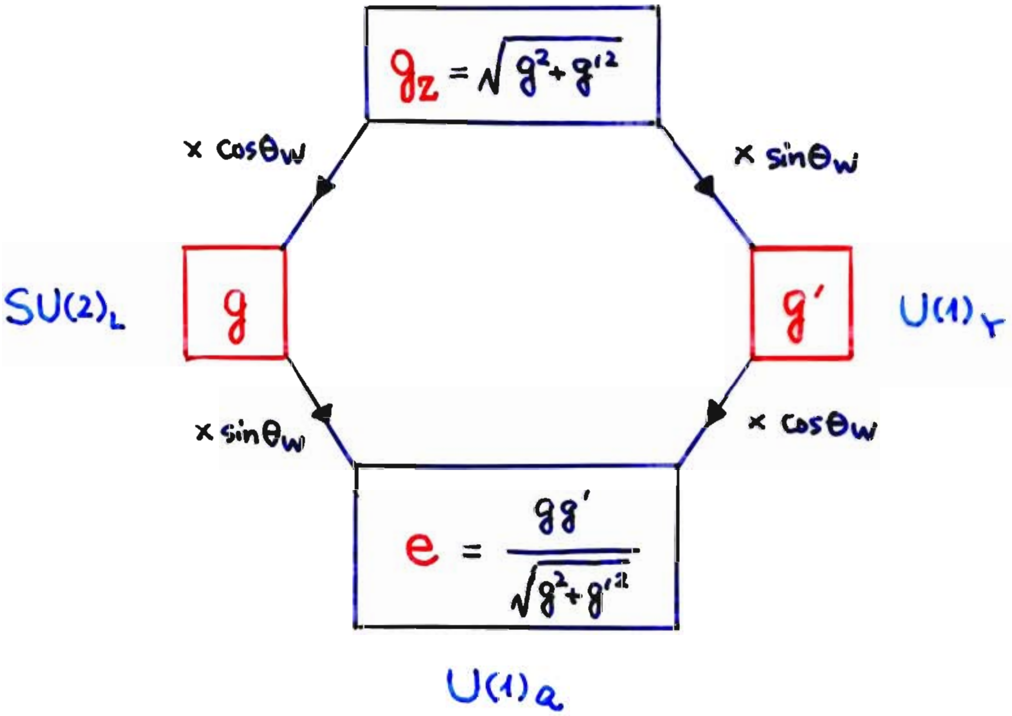
W_{T1}, W_{T2}, $\underbrace{W_L, W_S}_{\text{unphysical}}$, φ (+ghost)

broken

W_{T1}, W_{T2}, W_L, $\underbrace{W_S, \varphi}_{\text{unphysical}}$ (〃)

Weak mixing angle and gauge couplings
(Weinberg angle)

$$\boxed{\tan\theta_W = \frac{g'}{g}} \qquad \sin\theta_W = \frac{g'}{\sqrt{g^2+g'^2}} \qquad \cos\theta_W = \frac{g}{\sqrt{g^2+g'^2}}$$



ρ parameter

$$\rho \equiv \frac{m_W^2}{m_Z^2 \cos^2\theta_W}$$

$$= \frac{g_Z^2 / m_Z^2}{g^2 / m_W^2} = \frac{\text{NC Fermi coupling}}{\text{CC Fermi coupling}}$$

$= 1$ if Higgs doublet(s)

General Higgs

restriction for the breaking pattern (by hand)

ϕ Weak isospin I

VEVをもつ成分 I_3 $\quad Q = I_3 + Y = 0 \rightarrow Y = -I_3$

$$D_\mu \phi = \partial_\mu \phi + ig T^a W_\mu^a \phi - ig' I_3 B_\mu \phi$$

T^a: SU(2)の"I"表現

$$= \partial_\mu \phi + \frac{ig}{\sqrt{2}}(T^+ W_\mu + h.c.)\phi + i(gT^3 W_\mu^3 - g' I_3 B_\mu)\phi$$

$$m_W^2 = \frac{1}{2}\langle I_3 | T^+T^- + T^-T^+ | I_3 \rangle g^2 \langle \phi \rangle^2 \qquad T^\pm = T^1 \pm i T^2$$

$$= [\, I(I+1) - I_3^2 \,]\, g^2 \langle \phi \rangle^2$$

(If 複数のHiggs $\langle \phi \rangle^2 \rightarrow \sum_n \langle \phi_n \rangle^2$

$$m_Z^2 = 2 I_3^2 (g^2 + g'^2) \langle \phi \rangle^2$$

∘ $Z_\mu \propto g W_\mu^3 - g' B_\mu$ always

$$\boxed{\rho = \frac{m_W^2}{m_Z^2 \cos^2\theta_W} = \frac{I(I+1) - I_3^2}{2 I_3^2}}$$

(I, I_3)	ρ	
$(I, I_3) = (\frac{1}{2}, \pm\frac{1}{2})$	$\rho = 1$	← Experimentally confirmed
$(1, \pm 1)$	$\frac{1}{2}$	
$(1, 0)$	∞	(U(1)×U(1) unbroken)
$(\frac{3}{2}, \pm\frac{3}{2})$	$\frac{1}{3}$	
$(\frac{3}{2}, \pm\frac{1}{2})$	7	
⋮	⋮	
$(3, \pm 2)$	1	
$(\frac{25}{2}, \pm\frac{15}{2})$	1	

Always physical scalar(s)

▫ scalar場 (real n成分)の空間 : $\mathbb{R}^n$

→ always a radial mode ($\langle \phi \rangle$ の大きさを変える方向) : physical

▫ $SU(2) \times U(1) \rightarrow U(1)$ (3 broken generators)

$U(1)_Y$ を破るためには Higgsは nonzero U(1) charge をもつ → complex field

→ real成分は必ず偶数 → 偶数 − 3 ≠ 0

$SU(2) \times U(1)$ Gauge Interactions

$$D_\mu = \partial_\mu + igT^aW^a_\mu + ig'YB_\mu$$

$$= \partial_\mu + \frac{ig}{\sqrt{2}}\left(T^+W^\dagger_\mu + T^-W_\mu\right) + igT^3W^3_\mu + ig'YB_\mu$$

(W^3_μ, B_μ) part

introduce $Q = T_3 + Y$

(different conventions $Q = T_3 + \frac{1}{2}Y$, $Q = T_3 - Y$)

$$igT^3W^3_\mu + ig'(Q - T^3)B_\mu$$

$$= iT^3\underbrace{(gW^3_\mu - g'B_\mu)}_{g_Z Z_\mu} + ig'Q\underbrace{B_\mu}_{= -Z_\mu\sin\theta_W + A_\mu\cos\theta_W}$$

$$= ig_Z(T^3 - Q\sin^2\theta_W)Z_\mu + ieQA_\mu$$

$$\boxed{D_\mu = \partial_\mu + \frac{ig}{\sqrt{2}}\left(T^+W^\dagger_\mu + T^-W_\mu\right) + ig_Z(T^3 - Q\sin^2\theta_W)Z_\mu + ieQA_\mu}$$

$W^\pm$ couples with pure $SU(2)$ coupling

Z couples to a linear combination of $SU(2)$ and e.m. charge

photon couples to Q, coupling e

QED

$$T^\pm = T^1 \pm iT^2$$

$$g = \frac{e}{\sin\theta_W} \qquad g_Z = \frac{g}{\cos\theta_W} = \frac{e}{\sin\theta_W\cos\theta_W}$$

Fermion gauge interactions

$$\mathcal{L} = \underset{\substack{\text{all}\\\text{left-handed}\\\text{fields}}}{\sum} \bar{\psi}_L i \not{D} \psi_L \quad = \underset{\substack{\text{no}\\\text{antifermions}}}{\sum} \bar{\psi}_L i \not{D} \psi_L + \sum \bar{\psi}_R i \not{D} \psi_R$$

(if fermion number conserved)

Ordinary fermions

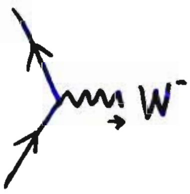
L : doublet

R : singlet

$$= \text{kinetic term} - \frac{g}{\sqrt{2}}\left(\bar{\psi}\gamma_\mu T^{+}\psi W_\mu^{\dagger} + \text{h.c.}\right)$$

$$- g_Z \bar{\psi}\gamma_\mu (T^3 - Q\sin^2\theta_W)\psi Z_\mu - e\bar{\psi}\gamma_\mu Q \psi A_\mu$$

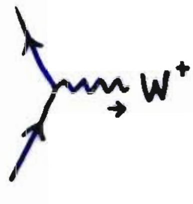
$(I_{3L} = +\frac{1}{2})$ $f_\uparrow$



$(I_{3L} = -\frac{1}{2})$ $f_\downarrow$

$$-\frac{ig}{\sqrt{2}}\gamma_\mu \frac{1-\gamma_5}{2}$$

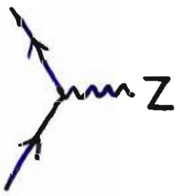
$f_\downarrow$



$f_\uparrow$

same

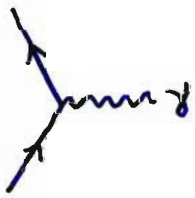
f



f

$$-ig_Z\gamma_\mu\left(I_3 \frac{1-\gamma_5}{2} - Q\sin^2\theta_W\right)$$

f



f

$$-ieQ\gamma_\mu$$

Scalar gauge interactions

$$\mathcal{L} = \sum D_\mu\phi^* D_\mu\phi \qquad \left(\tfrac{1}{2}(D_\mu\phi)^2 \text{ for real fields}\right)$$

g  derivative int.

g^2  seagull

For gauge-Higgs interactions, see later

Yang-Mills Interaction

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu \qquad a = 1, \cdots, \dim G$$

f^{abc} : gauge群の structure constant

$$[T^a, T^b] = i f^{abc} T^c$$

$$\mathrm{Tr}(T^a T^b) \propto \delta^{ab} \rightarrow f^{abc} : \text{totally antisymmetric}$$

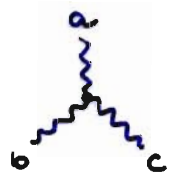
adjoint rep. : $(T^a)_{ij} = -i f^{aij}$ (f^{abc} 自身が表現. ∵ Jacobi id.)

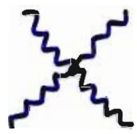
次元は群の次元に等しい.

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu}$$

$$= -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 + g f^{abc} \partial_\mu A^a_\nu A^b_\mu A^c_\nu - \frac{1}{4} g^2 f^{abe} f^{cde} A^a_\mu A^b_\nu A^c_\mu A^d_\nu$$

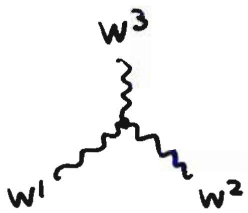
kinetic term





SU(2) $f^{abc} = \epsilon^{abc}$ $a = 1, 2, 3$

$$\epsilon^{abe}\epsilon^{cde} = \delta^{ac}\delta^{bd} - \delta^{ad}\delta^{bc}$$

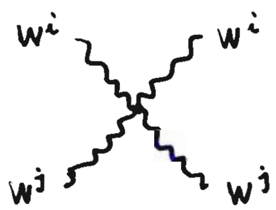


W^+W^-Z

$W^+W^-\gamma$

no ZZZ etc.

(all neutral)



$W^+W^-W^+W^-$

W^+W^-ZZ

$W^+W^-Z\gamma$

$W^+W^-\gamma\gamma$

no ZZZZ etc.

Fermion mass

Only Higgs doublets can give known fermion masses

($\leftrightarrow$ any nontrivial Higgs rep. can give gauge boson masses)

Quarks and leptons

		T	$Y = \langle Q \rangle$	$SU(3)_c$	invariant combination
ν, e^-	$l_L = \begin{pmatrix} \nu \\ e \end{pmatrix}_L$	$\frac{1}{2}$	$-\frac{1}{2}$	1	$l_L\, a_L\, \varphi^* + h.c.$ Y: $-\frac{1}{2}$, $+1$, $-\frac{1}{2}$
e^+	a_L	0	$+1$	1	
u, d	$q_L = \begin{pmatrix} u \\ d \end{pmatrix}_L$	$\frac{1}{2}$	$\frac{1}{6}$	3	$q_L\, n_L\, \varphi + h.c.$ Y: $\frac{1}{6}$, $-\frac{2}{3}$, $\frac{1}{2}$
$\bar{u}$	n_L	0	$-\frac{2}{3}$	3^*	
$\bar{d}$	p_L	0	$\frac{1}{3}$	3^*	$q_L\, p_L\, \varphi^* + h.c.$ Y: $\frac{1}{6}$, $\frac{1}{3}$, $-\frac{1}{2}$
	$\varphi = \begin{pmatrix} \varphi^+ \\ \varphi^0 \end{pmatrix}$	$\frac{1}{2}$	$\frac{1}{2}$	1	
	$\tilde{\varphi} = \begin{pmatrix} -\varphi^{0*} \\ \varphi^- \end{pmatrix}$	$\frac{1}{2}$	$-\frac{1}{2}$	1	

1. Fermions (above) cannot have $SU(2) \times U(1)$ invariant mass term

   Quarks/leptons are massless "before" symmetry breaking

2. A single doublet φ can generate masses of all quarks, leptons, and $W^\pm, Z$.

Quark masses and mixings (generic case)

n generations of quarks $\{ q^0_{Li}, u^0_{Ri}, d^0_{Ri} \}$ $i = 1, \cdots, n$

1. Kinetic + gauge term for these fields is symmetric under $U(n) \times U(n) \times U(n)$:

$$q^0_L \to U_1 q^0_L , \quad u^0_R \to U_2 u^0_R , \quad d^0_R \to U_3 d^0_R$$

- Gauge bosons are blind of generations $\quad q^0_L = \begin{pmatrix} q^0_{L1} \\ \vdots \\ q^0_{Ln} \end{pmatrix}$ etc.
- Every generation cannot be distinguished from one another

2. Yukawa interactions break the symmetry (generically)

$$-\mathcal{L} = f_{ij} \bar{d}^0_{Ri} \bar{\varphi} q^0_{Lj} + h_{ij} \bar{u}^0_{Ri} \tilde{\bar{\varphi}} q^0_{Lj} + \text{h.c.}$$

$$f_{ij}, h_{ij} : \text{complex}$$

- Yukawas are the only source of symmetry breaking in SM

3. f's and h's have $4n^2$ (real) parameters in total. But not all of them are physically observable.

The symmetry of the kinetic term implies a kind of reparametrization invariance : The transformation

$$f \to U_3^\dagger f U_1 , \quad h \to U_2^\dagger h U_1 \qquad \begin{matrix} f = (f_{ij}) \\ h = (h_{ij}) \end{matrix}$$

leaves the physics unchanged.

4. One should be careful at this point, because not the entire $U(n)^3$ symmetry really acts on f and h.

Actually a $U(1)$ subgroup of $U(n)^3$

$$U_1 = U_2 = U_3 = e^{i\alpha}$$

does not change f and h.

The effective reparametrization group is thus $U(n)^3/U(1)$.

5. The space of physical parameters is therefore

$$\mathbb{R}^{4n^2} / \left(U(n)^3/U(1) \right)$$

$$\text{Dimension} = 4n^2 - (3n^2 - 1) = n^2 + 1$$

This kind of analysis useful for restricted forms of Yukawas

$$n^2 + 1 = \underbrace{2n}_{\text{masses}} + \underbrace{\tfrac{1}{2}n(n-1)}_{\substack{\text{mixing angles} \\ \text{(flavor violation)}}} + \underbrace{\tfrac{1}{2}(n-1)(n-2)}_{\substack{\text{phases} \\ \text{(CP violation)}}}$$

	masses	mixing angles	phases
$n=2$	4	1	0
3	6	3	1
4	8	6	3

If one starts from real Yukawa couplings

\# total parameters $2n^2$

reparametrization group $O(n) \times O(n) \times O(n)$ dim $3 \times \frac{1}{2}n(n-1)$

(must keep the reality of f and h)

\# physical parameters $2n^2 - 3 \cdot \frac{1}{2}n(n-1) = \frac{1}{2}n(n+3)$

$$\frac{1}{2}n(n+3) = \underbrace{2n}_{\text{masses}} + \underbrace{\tfrac{1}{2}n(n-1)}_{\text{mixings}} \quad \text{(no phase)}$$

Quark mass matrix

$$M_d = \frac{v}{\sqrt{2}} f \qquad M_u = \frac{v}{\sqrt{2}} h$$

$$-\mathcal{L}_m = \bar{d}_R^0 M_d d_L^0 + \bar{u}_R^0 M_u u_L^0 + h.c. \qquad d_L^0 = \begin{pmatrix} d_{L1}^0 \\ \vdots \\ d_{Ln}^0 \end{pmatrix}$$

Theorem: a complex $n \times n$ matrix M can written as $\quad M = U^\dagger D U' \quad ; \quad U, U^\dagger$: unitary

D: diagonal, all elements ≥ 0

---

mass eigenstates

$$M_u = U_R^\dagger M_u^{diag.} U_L, \qquad M_d = V_R^\dagger M_d^{diag.} V_L$$

$$u_{\substack{R\\L}} \equiv U_{\substack{R\\L}} u_{\substack{R\\L}}^0, \quad d_{\substack{R\\L}} \equiv V_{\substack{R\\L}} d_{\substack{R\\L}}^0 \qquad \text{と定義すると}$$

$$-\mathcal{L}_m = \bar{d}_R M_d^{diag.} d_L + \bar{u}_R M_u^{diag.} u_L + h.c. \quad \text{は対角}$$

left-handed doublets

$SU(2)_L$ acts on $q_L^0 = \begin{pmatrix} u_L^0 \\ d_L^0 \end{pmatrix}$

or $\quad U_L q_L^0 = \begin{pmatrix} u_L \\ U_L V_L^\dagger d_L \end{pmatrix} \equiv \begin{pmatrix} u_L \\ d_L' \end{pmatrix}$

or $\quad V_L q_L^0 = \begin{pmatrix} V_L U_L^\dagger u_L \\ d_L \end{pmatrix} \equiv \begin{pmatrix} u_L' \\ d_L \end{pmatrix}$

$K \equiv U_L V_L^\dagger$: Kobayashi-Maskawa matrix

$$d_L' = K d_L \qquad u_L' = K^\dagger u_L$$

gauge coupling

Neutral : $\bar{u}_L \gamma_\mu u_L$ & $\bar{d}_L \gamma_\mu d_L$ $\qquad$ diagonal : GIM

Charged : $\bar{u}_L \gamma_\mu d_L' = \bar{u}_L \gamma_\mu K d_L$ $\qquad$ flavor violation

$SU(2) \times U(1) \times SU(3)$: Minimal model

(But dimensional transmutation)

The only dimensionful parameter is the Higgs mass term (or the Higgs VEV) : All other masses are secondary

(→ second kind of "universality" : see below)

$$m_W = \frac{1}{2} g v$$

$$m_Z = \frac{1}{2} g_Z v$$

$$m_u = \frac{1}{\sqrt{2}} h_u v$$

$$m_d = \frac{1}{\sqrt{2}} f_d v$$

$$m_\ell = \frac{1}{\sqrt{2}} f_\ell v$$

$$m_H = \sqrt{2} \lambda^{1/2} v$$

mass $\propto$ coupling with Higgs

Unique breaking pattern

Neutrinos exactly massless (Majorana mass terms forbidden)

Baryon and lepton number automatically conserved (no renormalizable violating interactions)

How neutrinos can get mass

1. Introducing ν_R → Dirac mass

ν_R : $SU(3) \times SU(2) \times U(1)$ singlet   no gauge coupling

$$-\mathcal{L}_m = h_\nu \bar{\nu}_R \tilde{\varphi}^\dagger \ell_L + \text{h.c.} \quad \rightarrow \quad m_\nu = \frac{1}{\sqrt{2}} h_\nu v$$

No explanation why $m_\nu \lll m_\ell, m_q$

2. Introducing triplet Higgs → Majorana mass

$\nu_L \cdot \nu_L \in \ell_L \cdot \ell_L$: $SU(2)$ triplet
$Y = -1$

need $\chi_a\,(T=1, Y=1)$ $(\chi^{++}, \chi^+, \chi^0)$

$$-\mathcal{L} = f_M\, \ell_L^T (-C^\dagger)(-i\tau_2)\tau_a \ell_L \chi_a + \text{h.c.} \rightarrow m_\nu = f_M \langle \chi^0 \rangle$$

vev $\langle \chi^0 \rangle$ must be much smaller than v ($\rho = 1$ constraint also)

- Can impose global lepton number symmetry (Gelmini-Roncadelli model)

  In this case $\langle \chi \rangle \neq 0$ → Goldstone boson (Majoron)

3. Remnant interactions from higher energy → Majorana mass

no new particles, nonrenormalizable effective interaction

$$-\mathcal{L}_{eff} = \frac{1}{M} \ell_L^T (-C^\dagger)(-i\tau_2)\tau_a \ell_L\, \varphi^T (-i\tau_2)\tau_a \varphi + \text{h.c.} \qquad \text{dim-5}$$

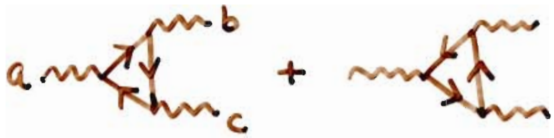
$$m_\nu \sim \frac{v^2}{M}$$

Small m_ν explained by large M

- "See-saw" : an explicit example   (ν_R with large Majorana mass + ordinary Dirac mass term)

Anomaly cancellation in SM

3-gauge boson vertex : fermion loop correction



一般に chiral fermion があると gauge current に anomaly が生じる
(→ くりこみ不可能)

$$\text{anomaly} \propto \sum_{\substack{\text{all}\\\text{left-handed}}} \mathrm{Tr}\, T^a \{T^b, T^c\}$$

または $\mathrm{Tr}_L - \mathrm{Tr}_R$
(without antiparticles)

Anomaly は fermion mass によらない

 ~ linear divergence ↔ anomaly

 mass = 0 (chirality)

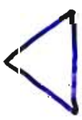
 ,  convergent

SU(2) × U(1)

① T^a  T^b T^c $=0$ SU(2) does not have complex reps

② Y  T^a T^b nontrivial

③ T^a  Y Y $=0$ $(\mathrm{Tr}\, T^a = 0)$

④ Y  Y Y nontrivial anomaly $\propto \mathrm{Tr}_L\, T^a\{T^b, T^c\} - \mathrm{Tr}_R\, T^a\{T^b, T^c\}$

Tr : for all fermions (no antifermions)

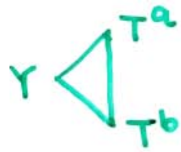
SM

(1) left-handed fermion は SU(2) doublet $T^a = \frac{1}{2}\tau_a$

(2) right-handed fermion は SU(2) singlet $T^a = 0$

(3) left と right の charge $Q = T^3 + Y$ は 等しい

② $\mathrm{Tr}_L\, Y \underbrace{\{T^a, T^b\}}_{(1)\ \frac{1}{2}\delta^{ab}\mathbb{1}} - \mathrm{Tr}_R\, Y \underbrace{\{T^a, T^b\}}_{(2)\ =0}$



$= \frac{1}{2}\delta^{ab}\,\mathrm{Tr}_L Y$

$= \frac{1}{2}\delta^{ab}\,\mathrm{Tr}_L (Q - T^3)$ → traceless

$= \frac{1}{2}\delta^{ab}\,\mathrm{Tr}_L Q$

④ $\mathrm{Tr}_L Y^3 - \mathrm{Tr}_R Y^3$

$= \mathrm{Tr}_L (Q - T^3)^3 - \mathrm{Tr}_R Q^3$ (2) ($T^3 = 0$ for right-handed fermions)

$= \cancel{\mathrm{Tr}_L Q^3} - 3\mathrm{Tr}_L Q^2 T^3 + 3\mathrm{Tr}_L Q (T^3)^2 - \underbrace{\mathrm{Tr}_L (T^3)^3}_{=0\ \mathrm{SU(2)}} - \underbrace{\cancel{\mathrm{Tr}_R Q^3}}_{(3)}$

$= -3\mathrm{Tr}_L (Q - T^3) Q T^3$

$= -3\mathrm{Tr}_L Y Q T^3$

$= -3\mathrm{Tr}_L Y (T^3 + Y) T^3$

$= -3\underbrace{\mathrm{Tr}_L Y (T^3)^2}_{=0} - 3\mathrm{Tr}_L Y^2 T^3$ → traceless

→ ② に帰着

$$\boxed{\text{Anomaly cancellation in SM} \leftrightarrow \sum Q = 0}$$

$Q_\nu + Q_e + \underset{\uparrow\ \text{color}}{3} Q_u + 3 Q_d = 0 + (-1) + 3 \times \frac{2}{3} + 3 \times (-\frac{1}{3}) = 0$

Anomaly

Simple group

既約表現 R $\quad \mathrm{Tr}\, T^a\{T^b, T^c\} = a(R)\, d^{abc}$

Conjugate rep. $\quad a(R^*) = -a(R)$

(Proof)
$$
\begin{aligned}
a(R^*)\, d^{abc} &= \mathrm{Tr}\,(-T^{a*})\{(-T^{b*}), (-T^{c*})\} \\
&= (-)^3\, \mathrm{Tr}\, T^{a*}(T^{b*}T^{c*} + T^{c*}T^{b*}) \\
&= -\,\mathrm{Tr}\,(\,\cdot\cdot\,)^T \\
&= -\,\mathrm{Tr}\,(T^cT^b + T^bT^c)\,T^a \qquad (T^\dagger = T) \\
&= -\,\mathrm{Tr}\, T^a\{T^b, T^c\} \\
&= -\,a(R)\, d^{abc}
\end{aligned}
$$

Real (pseudoreal 含む) rep. の anomaly は 0

if $R^* \sim R \quad a(R) = -a(R^*) = -a(R) = 0$

Complex rep. をもたない群は anomaly free (すべての表現がanomalyなし)

$SU(2)$, $SO(2n+1)$, $Sp(2n)$, $SO(4N)$,

E_7, E_8, F_4, G_2

Complex rep. をもつ群 $SU(N)\ (N\geq 3)$, $SO(4N+2)$, E_6

のうち $SO(4N+2)$ と E_6 は anomaly free

anomaly $\leftrightarrow$ 3次の Casimir

complex rep. $\leftrightarrow$ 奇数次の Casimir

(独立, lowest order)

Casimir の個数 = 群の rank

$D_n \sim SO(2n)$ の Casimir の次数

$2, 4, 6, \cdots, 2n-2\ ;\ n \quad \begin{cases} n = \text{even} \rightarrow \text{no complex rep.} \\ n = \text{odd} \rightarrow \text{complex rep. あり} \end{cases}$

$n=2$	$SO(4) \sim SU(2)\times SU(2)$	$2;\ 2$
$n=3$	$SO(6) \sim SU(4)$	$2, 4;\ 3$ $\rightarrow$ anomaly
$n=4$	$SO(8)$	$2, 4, 6;\ 4$
$n=5$	$SO(10)$	$2, 4, 6, 8;\ 5$

$SO(6)$ の anomaly $\quad \mathrm{Tr}\, M^{ij}\{M^{kl}, M^{mn}\} \propto \epsilon^{ijklmn}$ (only $SO(6)$!)

E_6 の Casimir の次数 $2, 5, 6, 8, 9, 12$: no anomaly

Simple group のうち SU(N) (N≥3) 以外 の ゲージ理論は
自動的に anomaly free になる.

ゲージ群が SU(N) (N≥3), U(1) を含む場合は フェルミオンの
表現が全体として anomaly = 0 となっていなければならない.

QCD では, フェルミオンは (quarks) $(3_L + 3_L^*) \times n_f$ で
全体として real な表現なので anomaly はない
(vectorlike)

SU(2) × U(1) : miraculous cancellation !

Bouchiat - Iliopoulos - Meyer

SU(5) GUT : フェルミオンは $(5_L^* + 10_L) \times n_g$

$5_L^* \to \bar{d}_L, \binom{\nu}{e^-}_L$, $10_L \to \binom{u}{d}_L, \bar{u}_L$, etc

$a(10) = a(5) = -a(5^*)$ → anomaly は cancel

SO(10) GUT : anomaly free

$$\underset{SO(10)}{16} = \underset{SU(5)}{5^* + 10} + \underset{\bar{\nu}_L}{1}$$

これにより, SU(5), ひいては SM の anomaly cancellation が
理解できる.

$(\text{Type})_{\text{rank}}$			$\dim G$	order of indep. Casimir (# = rank)
A_ℓ	$SU(\ell+1)$		$\ell(\ell+2)$	$2, 3, 4, \cdots, \ell+1$
B_ℓ	$SO(2\ell+1)$		$\ell(2\ell+1)$	$2, 4, 6, \cdots, 2\ell$
C_ℓ	$Sp(2\ell)$		$\ell(2\ell+1)$	$2, 4, 6, \cdots, 2\ell$
D_ℓ	$SO(2\ell)$	$\ell \geq 3$	$\ell(2\ell-1)$	$2, 4, 6, \cdots, 2\ell-2 ; \ell$
E_6			78	$2, 5, 6, 8, 9, 12$
E_7			133	$2, 6, 8, 10, 12, 14, 18$
E_8			248	$2, 8, 12, 14, 18, 20, 24, 30$
F_4			52	$2, 6, 8, 12$
G_2			14	$2, 6$

$A_1 \sim B_1 \sim C_1$
$B_2 \sim C_2$
$A_3 \sim D_3$